Progression

- **Direct (O(N^2))**
  - No data structures

- **Middleman: Degenerate Factorizations (O(N))**
  - Factorization needed, no data-structure
  - Factorization not always available

- **Pre-FMM: S-expansions and R-expansions (O(N^{3/2}))**
  - Factorizations in terms of near or far-field functions
  - O(N^{3/2}) for optimal number of boxes
  - Need boxes to organize source or target sets, and manage those pairs that require direct summation

- **SLFMM: S and R-expansions, S|R translation (O(N^{4/3}))**

- **MLFMM: S and R-expansions, S|S, S|R and R|R translations (O(N log N))**
Middleman Algorithm

Standard algorithm

Middleman algorithm

Sources

Evaluation Points

Sources

Evaluation Points

Total number of operations: $O(NM)$

Total number of operations: $O(N+M)$

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Idea of a Single Level FMM

Standard algorithm

Sources

Evaluation Points

\[ N \quad M \]

Total number of operations: \( O(NM) \)

SLFMM

Sources \( L \) groups

Evaluation Points \( K \) groups

\[ N \quad M \]

Total number of operations: \( O(N+M+KL) \)

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Why do we need SLFMM if Middleman has smaller complexity?

- Expansions can be valid in domains smaller than the computational domain.
- Even though expansion can be valid everywhere, the truncation number can be huge for large domains to provide accuracy.
- Sources and evaluation points can be spatially close, and there is a problem to evaluate singular potentials.
- Important theoretical question: determining optimal number of groups automatically

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Spatial Domains

Potentials due to sources in these spatial domains

$\Phi_1^{(n)}(y)$  $\Phi_2^{(n)}(y)$  $\Phi_3^{(n)}(y)$

$E_1$  $E_2$  $E_3$

$I_1(n) = n$  $I_2(n) = \{\text{Neighbors}(n)\} \cup n$  $I_3(n) = \{\text{All boxes}\} \setminus I_2(n)$

Boxes with these numbers belong to these spatial domains

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Definition of potentials

\[ \Phi^{(n)}_1(y) = \sum_{x_i \in E_1(n)} u_i \Phi(y, x_i), \]
\[ \Phi^{(n)}_2(y) = \sum_{x_i \in E_2(n)} u_i \Phi(y, x_i), \]
\[ \Phi^{(n)}_3(y) = \sum_{x_i \in E_3(n)} u_i \Phi(y, x_i), \]

Since domains \( E_2(n) \) and \( E_3(n) \) are complimentary:

\[ \Phi(y) = \sum_{i=1}^{N} u_i \Phi(y, x_i) = \sum_{x_i \in E_2(n) \cup E_3(n)} u_i \Phi(y, x_i) = \Phi^{(n)}_2(y) + \Phi^{(n)}_3(y), \]

for arbitrary \( n \).

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SLFMM Algorithm

Step 1. Generate S-expansion coefficients for each box

\[ \Phi^{(n)}_i(x) = C^{(n)} \circ S(x - x^{(n)}_c), \]
\[ C^{(n)} = \sum_{x_i \in E_1(n)} u_i B(x_i, x^{(n)}_c). \]

loop over all non-empty source boxes

For \( n \in \text{NonEmptySource} \)

Get \( x^{(n)}_c \), the center of the box;

\[ C^{(n)} = 0; \]

loop over all sources in the box

For \( x_i \in E_1(n) \)

Get \( B(x_i, x^{(n)}_c) \), the S-expansion coefficients near the center of the box;

\[ C^{(n)} = C^{(n)} + u_i B(x_i, x^{(n)}_c); \]

End;

End;

Implementation can be different!
All we need is to get \( C^{(n)}. \)

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SLFMM Algorithm

Step 2. (S|R)-translate expansion coefficients

\[
\Phi_3^{(n)}(y) = D^{(n)} \circ R(y - x_c^{(n)}),
\]

\[
D^{(n)} = \sum_{m \in I_3(n)} (S|R)(x_c^{(n)} - x_c^{(m)})C^{(m)}.
\]

For \( n \in \text{NonEmptyEvaluation} \)

Get \( x_c^{(n)} \), the center of the box;

\( D^{(n)} = 0; \)

For \( m \in I_3(n) \)

Get \( x_c^{(m)} \), the center of the box;

\( D^{(n)} = D^{(n)} + (S|R)(x_c^{(n)} - x_c^{(m)})C^{(m)}; \)

End;

End;

Implementation can be different!
All we need is to get \( D^{(n)}. \)

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S|R-translational
SLFMM Algorithm

Step 3. Final Summation

\[ v_j = \Phi(y_j) = \sum_{x_i \in E_2(n)} u_i \Phi(y_j, x_i) + D^{(n)} \circ R(y_j - x_c^{(n)}), \quad y_j \in E_1(n). \]

For \( n \in \text{NonEmptyEvaluation} \)

Get \( x_c^{(n)} \), the center of the box;

For \( y_j \in E_1(n) \)

\[ v_j = D^{(n)} \circ R(y_j - x_c^{(n)}); \]

For \( x_i \in E_2(n) \)

\[ v_j = v_j + \Phi(y_j, x_i); \]

End;

End;

End;

Implementation can be different!
All we need is to get \( v_j \)

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Asymptotic Complexity of SLFMM

- By some magic we can easily find neighbors, and lists of points in each box.
  - Assume that cost of finding these is less than the most expensive operation
  - Magic will come from data-structures
- Translation is performed by straightforward $P \times P$ matrix-vector multiplication, where $P(p)$ is the total length of the translation vector. So the complexity of a single translation is $O(P^2)$.
- The source and evaluation points are distributed uniformly, and there are $K$ boxes, with $s$ source points in each box ($s = N/K$). We call $s$ the grouping (or clustering) parameter.
- The number of neighbors for each box is $O(1)$.
Then Complexity is:

- For Step 1: \( O(PN) \)
- For Step 2: \( O(P^2K^2) \)
- For Step 3: \( O(PM+Ms) \)
- Total: \( O(PN + P^2K^2 + PM + Ms) = O(PN + P^2K^2 + PM + MN/K) \)
Selection of Optimal $K$ (or $s$)

\[
F(K) = PN + P^2K^2 + PM + PMN/K.
\]

\[
F''(K) = 2P^2K - PMN/K^2 = 0.
\]

\[
K_{opt} = \left( \frac{MN}{2P} \right)^{1/3} = O\left( \left( \frac{MN}{P} \right)^{1/3} \right).
\]

\[
s_{opt} = \frac{N}{K_{opt}} = \left( \frac{2PN^2}{M} \right)^{1/3} = O\left( \frac{PN^2}{M} \right)^{1/3}.
\]

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Complexity of Optimized SLFMM

\[
F(K_{opt}) = PN + P^2 \left( \frac{MN}{2P} \right)^{2/3} + PM + PMN \left( \frac{MN}{2P} \right)^{-1/3}
\]

\[
= P(M + N) + (MN)^{2/3} O(P^{4/3}).
\]

At \( K = K_{opt} \), and \( M = O(N) \), the complexity of SLFMM is:

\[
O(PN + P^{4/3} N^{4/3}) = O(P^{4/3} N^{4/3}).
\]
SLFMM Characteristics

- Group source points into clusters
  - Group evaluation points into clusters
- Find the cluster center \((x_*)\) for each cluster
- Find distances from cluster center to points in cluster
- Find distances between clusters
- Build a \(S\) representation for points in each source cluster
  \[
  \Phi(x_i, y) = \sum_{m=1}^{p} b_m(x_i, x_*) \cdot S_m (y - x_*)
  \]
  - Consolidate \(S\) series from all \(x_i\) in the cluster
  - For each evaluation box find clusters that are outside min. radius at which \(S\) expansion converges. Translate \(S\) series to \(R\) series
  - Evaluate \(R\) series at the evaluation points
  - Clean-up by evaluating at points which are inside min. radius

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Algorithms needed

- Space division and grouping of points
- Center finding for groups
- Distance between points (group centers)
- Neighbor finding
- Hierarchical division in FMM adds
  - Hierarchical grouping, center finding
  - Finding parents, children, siblings
  - Finding neighbors
- Spatial data-structures
- Also, the structures used to store these relationships require memory
- Determine optimal parameters and “break-even” point

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Spatial Data Structures

• Spatial data structures have developed since late 1980s,
  – Aside: one of the founders of the field (Hanan Samet) is local
• FMM developed over the same period
  – So don’t use the latest spatial data structure algorithms
• For effective “generalization” we need to use state-of-the-art spatial data structures
• Optimal data structures for FMM is an open research area
• In this course we will mostly use hierarchical division into boxes using $2^d$ trees
  – Use asymptotically optimal algorithms for performing operations on them

• Next class …

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• **Constant time methods to**
  – number the boxes in a way that can be generated from the coordinate
  – assign points to boxes using their binary coordinates
  – find box centers
  – Find neighboring boxes

• **Use bit interleaving and bit shift**

• **Apply to the general** $d$-**dimensional case**

• **Storage is also minimized as most necessary relations are directly generated from point coordinates**

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Summary of formal requirements for functions that can be used in FMM

- We have two sets of points:
  \[ X = \{x_1, x_2, \ldots, x_N\}, \quad x_i \in \mathbb{R}^d, \quad i = 1, \ldots, N, \]
  \[ Y = \{y_1, y_2, \ldots, y_M\}, \quad y_j \in \mathbb{R}^d, \quad j = 1, \ldots, M. \]

- We have functions (potentials):
  \[ \Phi(x_i, y) : \mathbb{R}^d \to \mathbb{R}, \quad y \in \mathbb{R}^d, \quad i = 1, \ldots, N. \]

- These functions can be factorized as (local expansion):
  \[ \Phi(x_i, y) = A(x_i, x_*) \circ R(y - x_*), \quad |y - x_*| < r < |x_i - x_*|, \quad i = 1, \ldots, N \]

- These functions can be factorized as (far field expansion):
  \[ \Phi(x_i, y) = B(x_i, x_*) \circ S(x - x_*), \quad |y - x_*| > R > |x_i - x_*|, \quad i = 1, \ldots, N \]

- The product is distributive operation with respect to addition
  \[ (u_1 A_1 + u_2 A_2) \circ F = u_1 A_1 \circ F + u_2 A_2 \circ F, \quad F = S, R \]
Summary of formal requirements for functions that can be used in FMM (2)

- $R$-expansion coefficients can be $R|R$-translated:
  \[ |x - x_{*2}| < |x_i - x_{*1}| - |x_{*1} - x_{*2}| : \]
  \[ A(x_i, x_{*2}) = (R|R)(x_{*2} - x_{*1})A(x_i, x_{*1}) \]

- $S$-expansion coefficients can be $S|S$-translated:
  \[ |x - x_{*2}| > |x_{*1} - x_{*2}| + |x_i - x_{*1}|, \]
  \[ B(x_i, x_{*2}) = (S|S)(x_{*2} - x_{*1})B(x_i, x_{*1}) \]

- $S$-expansion coefficients can be $S|R$-translated (converted to $R$-expansion coefficients)
  \[ |x - x_{*2}| < |x_{*1} - x_{*2}| + |x_i - x_{*1}|, \]
  \[ A(x_i, x_{*2}) = (S|R)(x_{*2} - x_{*1})B(x_i, x_{*1}) \]

- And we are looking for sums:
  \[ v_j = \sum_{i=1}^{N} u_i \Phi(y_j, x_i), \quad j = 1, \ldots, M. \]

- Some generalization are possible, say instead of $\Phi(y_j, x_i)$ we can consider $\Phi_i(y_j)$, etc.
p-Truncated Vectors

\[ \forall A \in \mathbb{R}^p, \exists \Phi^p(y) = \sum_{n=0}^{p-1} A_n F_n(y) \in \mathbb{F}^p(\Omega) \subset \mathbb{F}(\Omega). \]

\( \mathbb{F}^p(\Omega) \) is dense in \( \mathbb{F}(\Omega) \):

\[ \forall \Phi(y) \in \mathbb{F}(\Omega), \exists p, \Phi^p(y) \in \mathbb{F}^p(\Omega), \quad \| \Phi(y) - \Phi^p(y) \| = \min_{r \in \Omega} |\Phi(y) - \Phi^p(y)| < \varepsilon. \]
Matrix Representation of Linear Operators

Let $\Omega' \subset \Omega$ and $\mathcal{F}$ is a mapping of $\mathbb{F}(\Omega)$ to $\mathbb{F}(\Omega')$. Such mapping can be considered as action of operator $\mathcal{F}$ on $\Phi$:

$$\mathcal{F}[\Phi(y)] = \Phi(y), \quad \Phi(y) \in \mathbb{F}(\Omega), \quad \Phi(y) \in \mathbb{F}(\Omega') \subset \mathbb{F}(\Omega)$$

Respectively, operator $\mathcal{F}$ generates operator $\mathcal{F}$ that maps the space of expansion coefficients $\Lambda(\Omega) \rightarrow \Lambda(\Omega')$, which can be considered as representation of the operator $\mathcal{F}$ in the space of expansion coefficients:

$$\mathcal{F}A = \tilde{A}, \quad A \in \Lambda(\Omega), \quad \tilde{A} \in \Lambda(\Omega') \subset \Lambda(\Omega).$$

Inversely, if we introduce any transform of expansion coefficients $\mathcal{F}A = \tilde{A}$ which provides uniform convergence of function $\Phi(y)$ corresponding to these coefficients in $\Omega' \subset \Omega$ then such transform can be treated as operator $\mathcal{F}$ that convert one function from $\mathbb{F}(\Omega)$ to another.
p-Truncation (Projection) Operator

\[ \Pr(p)A = \tilde{A}, \quad A \in \Lambda(\Omega), \quad \tilde{A} \in \Lambda^p(\Omega). \]

\[
A = \begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_{p-1} \\
A_p \\
\vdots \\
A_{p+1}
\end{pmatrix} \rightarrow \tilde{A} = \begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_{p-1} \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix} A
\]

In space \( F(\Omega) \):

\[ \Pr(p)[\Phi(y)] = \Phi^p(y), \quad \Phi(y) \in F(\Omega), \quad \Phi^p(y) \in F^p(\Omega), \]

\[ \lim_{p \to \infty} \| \Phi(y) - \Pr(p)[\Phi(y)] \| = 0. \]
Norm of \( p \)-Truncation Operator (important for error bounds)

Norm:

\[
\|Pr(p)\| = \frac{\sup_{y \in \Omega} \|Pr(p)[\Phi(y)]\|}{\sup_{y \in \Omega} \|\Phi(y)\|}.
\]

Triangle inequality:

\[
\|I\| - \|I - Pr(p)\| \leq \|Pr(p)\| \leq \|I\| + \|I - Pr(p)\| = 1 + \|I - Pr(p)\|
\]

\[
\forall \epsilon > 0, \exists p, \|I - Pr(p)\| < \epsilon,
\]

so

\[
\forall \epsilon > 0, \exists p, 1 - \epsilon < \|Pr(p)\| < 1 + \epsilon,
\]
**p-Truncated Operator**

Let $H : F(\Omega) \to F(\Omega)$ be an operator, that is represented by infinite matrix

$$
H = \begin{pmatrix}
    h_{00} & h_{01} & \ldots & h_{0,p-1} & h_{0p} & \ldots \\
    h_{10} & h_{11} & \ldots & h_{1,p-1} & h_{1p} & \ldots \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
    h_{p-1,0} & h_{p-1,1} & \ldots & h_{p-1,p-1} & h_{p-1,p} & \ldots \\
    h_{p0} & h_{p1} & \ldots & h_{p,p-1} & h_{pp} & \ldots \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix},
$$

We call operator $H^{(p)} : F(\Omega) \to F(\Omega)$, $p$ - *truncated* if it is represented by matrix

$$
H^{(p)} = \begin{pmatrix}
    h_{00} & h_{01} & \ldots & h_{0,p-1} & 0 & \ldots \\
    h_{10} & h_{11} & \ldots & h_{1,p-1} & 0 & \ldots \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
    h_{p-1,0} & h_{p-1,1} & \ldots & h_{p-1,p-1} & 0 & \ldots \\
    0 & 0 & \ldots & 0 & 0 & \ldots \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix}
$$

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Norm of \( p \)-Truncated Operator
(important for error bounds)

**Theorem:** Let \( H : F(\Omega) \to F(\Omega) \), such that \( 0 < \| H \| < \infty \), and \( H^{(p)} : F(\Omega) \to F(\Omega) \) is the \( p \)-truncated operator \( H \). Let also \( p(\epsilon) \) be such that \( 1 - \epsilon < \| P_r(p) \| < 1 + \epsilon \). Then

\[
(1 - \epsilon)^2 < \| P_r(p) \|^2 = \frac{\| H^{(p)} \|}{\| H \|} = \| P_r(p) \|^2 < (1 + \epsilon)^2,
\]

\[
\lim_{p \to \infty} \frac{\| H^{(p)} \|}{\| H \|} = 1.
\]

**Proof.**
A \( p \)-truncated operator can be represented in the form

\[
H^{(p)} = P_r(p)H P_r(p)
\]

(check!)
So the norm of \( H^{(p)} \) is

\[
\| H^{(p)} \| = \| P_r(p) \| \| H \| \| P_r(p) \| = \| H \| \| P_r(p) \|^2.
\]

End of Proof.

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Translation Operator

Operator $T(t) : F(\Omega) \rightarrow F(\Omega')$, $\Omega' \subset \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is called translation operator corresponding to translation vector $t$, if

$$T(t)[\Phi(y)] = \Phi(y + t), \quad (y \in \Omega, \quad y + t \in \Omega').$$
Translation (Passive view point)

- Take a function or equivalently a function expressed as a series expansion and express it in another coordinate system (reference frame)
- Function is the same on the common parts of domains of definition
- but is represented in different forms
  - $\Phi(x, y)$
  - $\sum_m b_m(x, x_1) S_m(y-x_1)$
  - $\sum_m a_m(x, x_2) R_m(y-x_2)$ with $a_m = \mathcal{T}b_m$
Translation (Active view point)

- In a fixed coordinate system move the vector (or function), and evaluate it at the same point
- $\Phi(x_i,y) \rightarrow \Phi(x_i+t,y)$
- Functions evaluated at the same point are not the same
- Operator transforms the reference frame
Alibi and Alias points of view on translation operator

``Active” or “Alibi” point of view:
Operator transforms functions (vectors).
The reference frame does not change.

``Passive” or “Alias” point of view:
Functions (vectors) do not change.
Operator transforms the reference frame.

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• Translation operators of the two view-points are related to each other (transposes)
  – Active view point transforms the vectors (functions)
  – Passive view point transforms the coordinate frame (basis functions)

• What we want in the FMM is a way to express a vector with “coordinates” at one center to represent vector at another “center” and basis

• We use duality of view points to construct matrix version of our translation operators

• Express the basis function in new basis in terms of a series of old basis functions

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R|R-reexpansion

Let \( y - x_* \in \Omega_r(x_*) \subset \mathbb{R}^d \), \( \Omega_r(x_*) : |y - x_*| < r \), and \( \{ R_n(y - x_*) \} \) be a regular basis in \( C(\Omega) \). Let \( y - x_* + t \in \Omega_r(x_*) \) and

\[
R_n(y - x_* + t) = \sum_{l=0}^{\infty}(R|R)_l(t)R_l(y - x_*).
\]

Coefficients \( (R|R)_l(t) \) are called \( R|R - reexpansion coefficients \) (regular-to-regular), and infinite matrix

\[
(R|R)(t) = \begin{pmatrix}
(R|R)_{00} & (R|R)_{01} & \cdots \\
(R|R)_{10} & (R|R)_{11} & \cdots \\
\cdots & \cdots & \cdots
\end{pmatrix}
\]

is called \( R|R - reexpansion matrix \).

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Example of $R|R$-reexpansion

$$R_m(x) = x^m,$$

$$R_m(x+t) = (x+t)^m = x^m + \binom{m}{1}x^{m-1}t + \ldots + \binom{m}{m-1}xt^{m-1} + t^m$$

$$= \sum_{l=0}^{m} \binom{m}{l} t^l x^{m-l} = \sum_{l=0}^{m} \binom{m}{l} t^{m-l} x^l = \sum_{l=0}^{m} \binom{m}{l} t^{m-l} R_l(x),$$

$$(R|R)_{lm}(t) = \begin{cases} \binom{m}{l} t^{m-l}, & l \leq m, \\ 0, & l > m. \end{cases}$$

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R|R-translation operator

Translation operator $T(t)$ which is represented in regular basis $\{R_n(y - x_*)\}$ by the $R|R - reexpansion matrix$ is called $R|R$-translation operator.

$$T(t)[\Phi(y)] = \Phi(y + t)$$

$$(R|R)(t) = T(t).$$
Different translation operators

\[ T(t)[Φ(y)] = Φ(y + t) \]

The first letter shows the basis for \( Φ(y) \)

\[ T(t) = \begin{cases} 
(\mathcal{R}|\mathcal{R})(t) \\
(\mathcal{S}|\mathcal{S})(t) \\
(\mathcal{S}|\mathcal{R})(t) \\
(\mathcal{R}|\mathcal{S})(t) \end{cases} \]

The second letter shows the basis for \( Φ(y + t) \)

Needed only to show the expansion basis 
(for operator representation)
Matrix representation of $R|R$-translation operator

Consider

$$\Phi(y) = \sum_{n=0}^{\infty} A_n(x_*) R_n(y - x_*).$$

$$\Phi(y + t) = (R|R)(t)[\Phi(y)] = \sum_{n=0}^{\infty} A_n(x_*) (R|R)(t)[R_n(y - x_*)].$$

$$= \sum_{n=0}^{\infty} A_n(x_*) R_n(y - x_* + t)$$

$$= \sum_{n=0}^{\infty} A_n(x_*) \sum_{l=0}^{\infty} (R|R)_l(t) R_l(y - x_*)$$

$$= \sum_{l=0}^{\infty} \left[ \sum_{n=0}^{\infty} (R|R)_l(t) A_n(x_*) \right] R_l(y - x_*)$$

$$= \sum_{l=0}^{\infty} \tilde{A}_l(x_*, t) R_l(y - x_*),$$

$$\tilde{A}_l(x_*, t) = \sum_{n=0}^{\infty} (R|R)_l(t) A_n(x_*) \quad \tilde{A}(x_*, t) = (R|R)(t) \tilde{A}(x_*).$$

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Reexpansion of the same function over shifted basis

Compact notation:

\[
\Phi(y) = \sum_{n=0}^{\infty} A_n(x_*) R_n(y - x_*) = A(x_*) \circ R(y - x_*)
\]

\[
\Phi(y + t) = \sum_{l=0}^{\infty} \tilde{A}_l(x_*, t) R_l(y - x_*) = \tilde{A}(x_*, t) \circ R(y - x_*)
\]

We have:

\[
\Phi(y) = \Phi((y - t) + t) = \tilde{A}(x_*, t) \circ R((y - t) - x_*)
\]

\[
= \tilde{A}(x_*, t) \circ R(y - x_* - t).
\]

Also

\[
\Phi(y) = A(x_*) \circ R(y - x_*) = A(x_* + t) \circ R(y - x_* - t),
\]

so

\[
A(x_* + t) = \tilde{A}(x_*, t) = (R|R)(t) A(x_*).
\]
\( R|R \)-reexpansion of the same function over shifted basis (2)

\[ \Omega_{r_1}(x_*+t) \]

Original expansion
Is valid only here!

\[ |y - x_* - t| < r_1 = r - |t| \]

Since \( \Omega_{r_1}(x_*+t) \subset \Omega_r(t) \)!
Example of power series reexpansion

\[ R_m(x) = x^m; \]

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} A_m(x_{*1}, x_i) R_m(y - x_{*1}) = \sum_{m=0}^{\infty} A_m(x_{*2}, x_i) R_m(y - x_{*2}), \]

\[ A(x_{*2}, x_i) = (R|R)(x_{*2} - x_{*1}) \cdot A(x_{*1}, x_i). \]

\[
\begin{pmatrix}
A_0(x_{*2}, x_i) \\
A_1(x_{*2}, x_i) \\
A_2(x_{*2}, x_i) \\
... \\
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\
1 & 0 \\
2 & 0 \\
2 & 1 \\
2 & 1 \\
... & ... \\
... & ... \\
1 & ... \\
... & ... \\
\end{pmatrix} \begin{pmatrix} A_0(x_{*1}, x_i) \\
A_1(x_{*1}, x_i) \\
A_2(x_{*1}, x_i) \\
... \\
\end{pmatrix}
\]

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Example of power series reexpansion (2).
Relation to Taylor series.

Let’s check this for Taylor series, when expansion coefficients are

\[ A_m(x_{*1},x_{i}) = \frac{1}{m!} \frac{\partial^m \Phi(x_{*1},x_{i})}{\partial x_{*1}^m} \]

For \( A_0 \) this yields Taylor series again!

Check for \( A_l \)

\[ \Phi(x_{*2},x_{i}) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1},x_{i})}{\partial x_{*1}^m} (x_{*2} - x_{*1})^m, \]

\[ \frac{1}{l!} \frac{\partial^l \Phi(x_{*2},x_{i})}{\partial x_{*2}^l} = \sum_{m=l}^{\infty} \left( \begin{array}{c} m \\ l \end{array} \right) \frac{1}{m!} \frac{\partial^m \Phi(x_{*1},x_{i})}{\partial x_{*1}^m} (x_{*2} - x_{*1})^{m-l} \]

\[ = \sum_{m=l}^{\infty} \frac{m!}{l!(m-l)!} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1},x_{i})}{\partial x_{*1}^m} (x_{*2} - x_{*1})^{m-l} \]

\[ = \frac{1}{l!} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{k+l} \Phi(x_{*1},x_{i})}{\partial x_{*1}^{k+l}} (x_{*2} - x_{*1})^k, \]

\[ \frac{\partial^l \Phi(x_{*2},x_{i})}{\partial x_{*2}^l} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{k+l} \Phi(x_{*1},x_{i})}{\partial x_{*1}^{k+l}} (x_{*2} - x_{*1})^k. \]

For \( A_l \) we obtained Taylor series for the \( l \)-th derivative! Wow!

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S|S-reexpansion

Let \( y - x_\ast \in \Omega_r(x_\ast) \subset \mathbb{R}^d \), \( \Omega_r(x_\ast) : |y - x_\ast| > r \), and \( \{S_n(y - x_\ast)\} \) be a singular basis in \( C(\Omega) \). Let \( y - x_\ast + t \in \Omega_r(x_\ast) \) and

\[
S_n(y - x_\ast + t) = \sum_{l=0}^{\infty} (S|S)_{ln}(t)S_l(y - x_\ast).
\]

Coefficients \( (S|S)_{ln}(t) \) are called \( S|S-reexpansion coefficients \) (singular-to-singular), and infinite matrix

\[
(S|S)(t) = \begin{pmatrix}
(S|S)_{00} & (S|S)_{01} & \cdots \\
(S|S)_{10} & (S|S)_{11} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

is called \( S|S-reexpansion matrix \).

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S|S-translation operator

Translation operator $T(t)$ which is represented in singular basis $\{S_n(y - x_\star)\}$ by the $S|S$ - reexpansion matrix is called $S|S$-translation operator.

$$T(t)[\Phi(y)] = \Phi(y + t)$$

$$(S|S)(t) = T(t).$$
S|R-translation operator

Translation operator $T(t)$ which is represented in singular basis by the $S|R$ – reexpansion matrix is called $S|R$-translation operator if the basis of expansion is changed with the translation operation from singular $\{S_n(y - x_*)\}$ to regular $\{R_n(y - x_* + t)\}$

$$T(t)[\Phi(y)] = \Phi(y + t)$$

$$(S|R)(t) = T(t).$$
S|R-operator has almost the same properties as S|S and R|R

(t cannot be zero)

\[ \Phi(y) = B(x_*) \circ S(y - x_*) , \]

\[ \Phi(y + t) = \tilde{A}(x_*, t) \circ R(y - x_*) \]

\[ \Phi(y) = \tilde{A}(x_*, t) \circ R(y - x_* - t) . \]

\[ \tilde{A}(x_*, t) = (S|R)(t)B(x_*) . \]
Picture is different…

Original expansion
Is valid only here!

\[ |y - x_* - t| < r_1 = |t| - r \]

Since
\( \Omega_{r_1}(x_*+t) \subset \Omega_r(t) \)!

Also
\[ |x_i - x_*| < r \]
singular point!
Example from previous lectures

\[ \Phi(y, x_i) = \frac{1}{y - x_i}. \]

\[ |y - x_*| < |x_i - x_*| : \quad \text{R-expansion} \]

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y - x_*), \]

\[ a_m(x_i, x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, ..., \]

\[ R_m(y - x_*) = (y - x_*)^m, \quad m = 0, 1, ... \]

\[ |y - x_*| > |x_i - x_*| : \quad \text{S-expansion} \]

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*), \]

\[ b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, ..., \]

\[ S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, ... \]
In this case we have

\[ (|y-x_\star| < |t|) \]

\[ S_n(y-x_\star+t) = (y+x)^{n-1} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} (y-x_\star)^m \]

\[ = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} R_m(y-x_\star) = \sum_{m=0}^{\infty} (S|R)_{mn}(t) R_m(y-x_\star). \]

So

\[ (S|R)_{mn}(t) = \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} = \frac{(-1)^m(m+n)!}{m!n!(n+m+1)!} \]

\[ (S|R)(t) = \begin{pmatrix} t^{-1} & t^{-2} & t^{-3} & \cdots \\ -t^{-2} & -2t^{-3} & -3t^{-4} & \cdots \\ t^{-3} & 3t^{-4} & 6t^{-5} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \]

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Norm of the Translation Operator

**Theorem.** Let \( F(\Omega) \) be a set of functions bounded in \( \mathbb{R}^d \). Then \( \| T(t) \| = 1 \).

**Proof.**

\[
\| T(t) \| = \frac{\| T(t) \Phi(y) \|}{\| \Phi(y) \|} = \frac{\| \Phi(y + t) \|}{\| \Phi(y) \|} = \frac{\sup_{y \in \mathbb{R}^d} |\Phi(y + t)|}{\sup_{y \in \mathbb{R}^d} |\Phi(y)|} = 1.
\]

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Norms of $R|R$, $S|S$, and $S|R$-operators (1)

\[ T(t) \]
\[ \Omega \]
\[ \Omega' \]
\[ \Phi(y) \]

\[ \begin{align*}
\Phi(y) \text{ is bounded in } \Omega. \\
\Omega' \subset \Omega. \\
\text{Therefore } \Phi(y) \text{ is bounded in } \Omega', \text{ and}
\end{align*} \]

\[ \| \Phi(y) \|_{\Omega'} = \sup_{y \in \Omega'} |\Phi(y)| \leq \sup_{y \in \Omega} |\Phi(y)| = \| \Phi(y) \|_{\Omega}. \]

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Norms of $R|R$, $S|S$, and $S|R$-operators (2)

From the passive point of view, the translation operator does nothing, but just changes the reference frame. So if we consider that $R|R$, $S|S$, and $S|R$ do just change of the reference frame **PLUS they shrink the domain, where the function is bounded, then their norms do not exceed** 1.

\[
\Omega' \subset \Omega
\]

\[
\| (R|R)(t) \| = \frac{\sup_{y \in \Omega'} |\Phi(y)|}{\sup_{y \in \Omega} |\Phi(y)|} \leq 1,
\]

\[
\| (S|S)(t) \| = \frac{\sup_{y \in \Omega'} |\Phi(y)|}{\sup_{y \in \Omega} |\Phi(y)|} \leq 1,
\]

\[
\| (S|R)(t) \| = \frac{\sup_{y \in \Omega'} |\Phi(y)|}{\sup_{y \in \Omega} |\Phi(y)|} \leq 1.
\]

This is the difference between general translation operator and $R|R$, $S|S$, and $S|R$ operators.
Error of exact $R|R$, $S|S$, and $S|R$-translation

If

$$\| \Phi(y) - \Phi^p(y) \| < \epsilon,$$

then

$$\| (R|R)(t)(\Phi(y) - \Phi^p(y)) \| = \| (R|R)(t) \| \| \Phi(y) - \Phi^p(y) \| < \epsilon,$$

$$\| (S|S)(t)(\Phi(y) - \Phi^p(y)) \| = \| (S|S)(t) \| \| \Phi(y) - \Phi^p(y) \| < \epsilon,$$

$$\| (S|R)(t)(\Phi(y) - \Phi^p(y)) \| = \| (S|R)(t) \| \| \Phi(y) - \Phi^p(y) \| < \epsilon.$$

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Five Key Stones of FMM

• Factorization
• Error
• Translation
• Grouping
• Data Structure

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Group Theory

- One of the key research areas in the FMM is reducing the cost of translations
- If one finds symmetries then computational costs are reduced
- Formal mathematics to study symmetry is Group Theory
- Interestingly FMM translational operators form a group
- Research: Use GT to improve efficiency

A set $G$ of elements (objects) $a, b, c, \ldots$ is called group if there defined some binary operation $\odot$ called “group operation”, which for any pair of elements $a, b \in G$ correspond some object $a \odot b$, such that for any $a, b, c \in G$ satisfies the following properties:

1). $a \odot b \in G$,  $(G$ is closed with respect to $\odot$),
2). $a \odot (b \odot c) = (a \odot b) \odot c$,  $(associativity)$,
3). $\exists e \in G$, $e \odot a = a$,  $(G$ contains the unity, $e$),
4). $\exists a^{-1} \in G$, $a^{-1} \odot a = e$,  $(G$ contains the inverse element for each $a \in G$).

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