1 Homework 5 (Solution)

1. Given

\[
\frac{1}{y - x_i} = \frac{1}{(y - x) - (x_i - x)} = \frac{1}{y - x} \left(1 - \frac{x_i - x}{y - x} \right)^{-1}
\]

Following the hint we can write

\[
\frac{1}{y - x_i} = \frac{1}{y - x} \left(1 + \frac{x_i - x}{y - x} + \ldots + \frac{(x_i - x)^{p-1}}{(y - x)^{(p-1)}} \right) + \frac{1}{y - x} \left(\frac{x_i - x}{y - x}\right)^p \frac{1}{1 - \frac{x_i - x}{y - x}}
\]

The required S expansion is then

\[
\Phi(x_i, y) = \frac{1}{y - x_i} = \sum_{m=0}^{p-1} b_m(x_i, x) S_m(y - x) + \text{Error}(p)
\]

with

\[
b_m(x_i, x) = (x_i - x)^m \quad \text{and} \quad S_m(y - x) = (y - x)^{-m-1},
\]

and the residual term is given by

\[
\text{Error}(p) = \frac{1}{y - x} \left(\frac{x_i - x}{y - x}\right)^p \frac{1}{1 - \frac{x_i - x}{y - x}} = \frac{(x_i - x)^p}{(y - x)^p} \frac{1}{y - x_i}
\]

We are given \( x_i = 0 \). So the terms can be written as

\[
b_m(x_i, x) = x_i^m, \quad S_m(y - x) = y^{-m}, \quad \text{Error}(p) = \frac{x_i^p}{y^p} \frac{1}{\frac{3}{2}l - x_i}
\]

Our goal is now to evaluate a relationship between the maximum magnitude of the residual, and \( p \) and \( l \). We are given \( x_i \in \left[-\frac{l}{2}, \frac{l}{2}\right], \ y \in \left[\frac{3}{2}l, \frac{5}{2}l\right] \). Looking at the error function we see that, given that \( y \) is positive increasing \( y \) should decrease the function monotonically. Because we wish to be conservative in our error estimates we can set \( y = \frac{3}{2}l \) and obtain

\[
\text{Error}(p) \leq \frac{x_i^p}{\left(\frac{3}{2}l\right)^p} \frac{1}{\frac{3}{2}l - x_i}
\]

The situation with \( x \) is a bit more complex as \( x \) has both positive and negative values. We can find the minimum of the function of \( p \) with \( x_i \)

\[
\frac{\partial E}{\partial x_i} = \frac{px_i^{p-1}}{y^p} \frac{1}{y - x_i} + \frac{x_i^p}{y^p} \left(-\frac{1}{(y - x_i)^2}\right) (-1) = \frac{px_i^{p-1}}{y^p} \frac{1}{y - x_i} + \frac{x_i^p}{y^p} \left(\frac{1}{y - x_i}\right)^2
\]

\[
= \frac{x_i^{p-1}}{y^p (y - x_i)} \left(p + \frac{x_i}{y - x_i}\right) = \frac{x_i^{p-1}}{\left(\frac{3}{2}l - x_i\right)} \left(p + \frac{x_i}{\frac{3}{2}l - x_i}\right) \left(p + \frac{x_i}{\frac{3}{2}l - x_i}\right)
\]

This vanishes when

\[
x_i = -p \left(\frac{3}{2}l - x_i\right)
\]

or

\[
(p - 1) x_i = \frac{3p}{2}l, \quad \text{or} \quad x_i = \frac{3p}{2(p - 1)}l
\]
Figure 1:

For large $p$ this is close to $1.5l$ while at $p = 2$ it is at $3l$, and so the extremum is attained outside the range $[-\frac{l}{2}, \frac{l}{2}]$. We can therefore bound the error by choosing the largest value of the numerator and the smallest one of the denominator

$$E_S(p) \leq \left(\frac{1}{2}\right)^p \frac{lp}{l} \frac{1}{2} = \frac{1}{3lp} = \epsilon.$$ 

So we have

$$\log \epsilon = -p \log 3 - \log l,$$

or

$$p = \left\lceil \frac{-\log \epsilon + \log l}{\log 3} \right\rceil = \left\lceil \log_3 \frac{1}{\epsilon} \right\rceil,$$

and $p$ can be chosen as the nearest integer above this estimate.
Figure 2:

- Theory
- Experiment, $x=0.5l$, $y=1.5l$

$p=10$

Max abs error, $\epsilon$

The length of interval, $l$
Figure 3:

Experiment, \( x_i \) and \( y \) are random points from \([0.5^*, 0.5^*]\) and \([1.5^*, 2.5^*]\), respectively.
Figure 4:

The length of interval, $l$

Max abs error, $\varepsilon$

Experiment, $x_i$ and $y$ are random points from $[-0.5^*,0.5^*]$ and $[1.5^*,2.5^*]$, respectively

Theory, $\rho=10$
2. The $R$ expansion is for $(y - x_*) < (x_i - x_*)$, and is given by

\[
\frac{1}{y - x_i} = \frac{1}{(y - x_*) - (x_i - x_*)} = \frac{1}{-(x_i - x_*)} \left(1 - \frac{y - x_*}{x_i - x_*}\right)
\]

\[
= \frac{-1}{(x_i - x_*)} \left(1 + \frac{y - x_*}{x_i - x_*} + \ldots + \left(\frac{y - x_*}{x_i - x_*}\right)^{p-1}\right) - \frac{1}{(x_i - x_*)} \frac{1}{1 - \frac{y - x_*}{x_i - x_*}} \left(\frac{y - x_*}{x_i - x_*}\right)^p
\]

\[
a_m(x, x_*) = -(x_i - x_*)^{-m-1} \quad R_m(y - x_*) = (y - x_*)^m \quad \text{Error}(p) = \frac{1}{y - x_i} \left(\frac{y - x_*}{x_i - x_*}\right)^p
\]

For doing the $p$-truncated $S|R$-translation of $S$-expansion coefficients to $R$-expansion coefficients we use the expression for the matrix given in class

\[
(S|R)_{mn}(t) = 1
\]

\[
\frac{d^m S_n(t)}{dt^m} = \frac{(-1)^m (m+n)!}{m!n!t^{n+m+1}}
\]

\[
(S|R)(t) = \begin{pmatrix}
 t^{-1} & t^{-2} & t^{-3} & \ldots \\
 -t^{-2} & -2t^{-3} & -3t^{-4} & \ldots \\
 t^{-3} & 3t^{-4} & 6t^{-5} & \ldots \\
 \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

A matlab function to calculate this matrix and return the $R$ coefficients after a matrix vector product is attached.

3. For $l = 2$ we show the results of the $p = 20$ truncated $S$-expansion of $\Phi(y, x_i)$ shifted by the vector $t = 4$. Convolve the output vector of $R$-expansion coefficients with $R$-basis functions centered at $x_* = 2l$ to get approximate value of $\Phi(y, x_i)$ at $y \in \left[\frac{2}{2}, \frac{5}{2}\right]$. Compare this result with exact (straightforward) computation of $\Phi(y, x_i)$.

translation is performed exactly we have the following series:

\[
\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y-x_*) ,
\]

\[
a_m(x_i, x_*) = -(x_i - x_*)^{-m-1} , \ m = 0, 1, ...
\]

\[
R_m(y-x_*) = (y-x_*)^m , \ m = 0, 1, ...
\]

which should be considered near \( x_* = 2t \). Truncation error of the exact expansion is

\[
\left| \Phi(y, x_i) - \sum_{m=0}^{p-1} a_m(x_i, x_*) R_m(y-x_*) \right| = \left| \sum_{m=p}^{\infty} a_m(x_i, x_*) R_m(y-x_*) \right| = \left| - \frac{1}{x_i - x_*} \sum_{m=p}^{\infty} \frac{(y-x_*)^m}{x_i - x_*} \right| = \left| \frac{1}{y-x_i} \left( \frac{y-x_*}{x_i - x_*} \right)^p \right| \leq \frac{1}{l} \frac{1}{3^p}.
\]

This shows that the error of the \( p \)-truncation operator both for \( S^- \) and \( R^- \) series is the same, so we have

\[
|Pr(p)\Phi(y, x_i) - \Phi(y, x_i)| \leq \epsilon, \ \epsilon = \frac{1}{l} \frac{1}{3^p}.
\]

So the norm of the truncation operation, \( Pr(p) \) does not exceed \( 1 + \epsilon \). Also \( \|T(t)\| \leq 1 \), so

\[
\|Pr(p)\| \|T(t)\| \leq 1 + \epsilon.
\]

Consider now the error of the \( p \)-truncated translation:

\[
\left| T^{(p)}(t)\Phi(y, x_i) - T(t)\Phi(y, x_i) \right| = |Pr(p)T(t) Pr(p)\Phi(y, x_i) - T(t)\Phi(y, x_i)| = |Pr(p)T(t) Pr(p)\Phi(y, x_i) - Pr(p)T(t)\Phi(y, x_i) + Pr(p)T(t)\Phi(y, x_i) - T(t)\Phi(y, x_i)| \\
\leq |Pr(p)T(t) Pr(p)\Phi(y, x_i) - Pr(p)T(t)\Phi(y, x_i)| + |Pr(p)T(t)\Phi(y, x_i) - T(t)\Phi(y, x_i)| \\
\leq |Pr(p)T(t)| |Pr(p)\Phi(y, x_i) - \Phi(y, x_i)| + |Pr(p)\Phi(y, x_i) - \Phi(y, x_i)| \\
\leq (1 + \epsilon)\epsilon + \epsilon = 2\epsilon + \epsilon^2 \approx 2\epsilon.
\]

4. **Method 2. Tighter Error Bound. Direct Evaluation.** Consider S-expansion near point \( x_{s1} \) (in our case \( x_{s1} = 0 \)):

\[
\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_{s1}) S_m(y-x_{s1}) ,
\]

\[
b_m(x_i, x_{s1}) = (x_i - x_{s1})^m , \ m = 0, 1, ...
\]

\[
S_m(y-x_{s1}) = (y-x_{s1})^{-m-1} , \ m = 0, 1, ...
\]

Exact S|R-translation to new expansion center \( x_{s2} \) (in our case \( x_{s2} = 2t = t \)) can be performed using infinite matrix

\[
(S|R)_{mn}(t) = \frac{(-1)^m(m+n)!}{m!n!(m+n+1)} , \ t = x_{s2} - x_{s1}
\]

so

\[
a_m(x_i, x_{s2}) = \sum_{n=0}^{\infty} (S|R)_{mn}(t) b_n(x_i, x_{s1}).
\]
In this case we have
\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn}, \]
where
\[ c_{mn} = (S|R)_{mn}(t) b_n(x_i, x_{*1}) R_m(y - x_{*2}) \]
\[ = \frac{(-1)^m(m + n)!}{m!n!t^{m+n+1}} (x_i - x_{*1})^m (y - x_{*2})^n \]
Translation with p–truncated operator \((S|R)^{(p)}_{mn}(t)\) yields
\[ \Phi^{(p)}(y, x_i) = \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn}. \]
The error of truncated translation is therefore
\[
\left| \Phi(y, x_i) - \Phi^{(p)}(y, x_i) \right| = \left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} - \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn} \right|
\]
\[ = \left| \sum_{m=0}^{p-1} \sum_{n=0}^{\infty} c_{mn} + \sum_{m=0}^{\infty} \sum_{n=0}^{p-1} c_{mn} - \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn} \right|
\]
\[ = \left| \sum_{m=0}^{p-1} \sum_{n=0}^{\infty} c_{mn} + \sum_{m=0}^{\infty} \sum_{n=0}^{p-1} c_{mn} \right| \leq \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} |c_{mn}| \leq \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} |c_{mn}|
\]
\[ \leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_{mn}|
\]
\[ \leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m + n)!}{m!n!t^{m+n+1}} \left| [x_i - x_{*1}]^m |y - x_{*2}|^n + [x_i - x_{*1}]^m |y - x_{*2}|^n \right|
\]
\[ \leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m + n)!}{m!n!t^{m+n+1}} \left[ \frac{1}{2^n} \frac{1}{2^m} \frac{1}{2^n} \right]
\]
\[ = 2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m + n)!}{m!n!} \left( \frac{1}{4} \right)^{m+n} + 2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m + n)!}{m!n!} \left( \frac{1}{2} \right)^{m+n+1}
\]
\[ = \frac{1}{7} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m + n)!}{m!n!} \left( \frac{1}{4} \right)^{n+m} = \frac{1}{7} \sum_{m=0}^{\infty} \frac{1}{4} \sum_{n=0}^{m} \frac{(m + n)!}{m!n!} \left( \frac{1}{4} \right)^{n+m+n}
\]
\[ = \frac{1}{7} \sum_{m=0}^{\infty} \frac{1}{4} \left( \frac{1}{4} \right)^{m+1} \left( \frac{1}{4} \right)^{m+1} = \frac{1}{7} \sum_{m=0}^{\infty} \frac{1}{3^{m+1}} = \frac{4}{7} \sum_{m=0}^{\infty} \frac{1}{3^{m+1}} = \frac{4}{7} \sum_{l=0}^{\infty} \frac{1}{3^l} = \frac{4}{7} \frac{1}{3^l} \frac{1}{3} = \frac{4}{7} \sum_{l=0}^{\infty} \frac{1}{3^{l+1}}
\]
\[ = \frac{2}{7} \sum_{m=0}^{\infty} \frac{1}{3^{m+1}} = \frac{2}{7} \sum_{m=0}^{\infty} \frac{1}{3^m} = 2 \epsilon, \quad (\epsilon = \frac{1}{7} \frac{1}{3^p}).
Here we used the fact that

\[
\frac{1}{(1 - \alpha)^{m+1}} = 1 + (m + 1) \alpha + \frac{(m + 1)(m + 2)}{2!} \alpha^2 + \ldots = \sum_{n=0}^{\infty} \frac{(m + n)!}{m!n!} \alpha^n, \quad |\alpha| < 1.
\]
Figure 6:

- Error of using truncated translation operator:
  \( x = 0.5^*l, \ y = 1.5^*l \)

- Single truncation error:
  \( x = 0.5^*l, \ y = 1.5^*l \)

- Theory
- Experiment

The length of the interval, \( l \)

Max abs error, \( \epsilon \)
Figure 7:

- Method 1
- Method 2
- Theory
- Error of truncated translation
- Max abs error, $\epsilon$
- Truncation number, $p$
- Single truncation error
- $x_i$ and $y$ are random points on intervals $[-0.5^\circ, 0.5^\circ]$ and $[1.5^\circ, 2.5^\circ]$, respectively.
Figure 8: