1. (50 points) Be concise in answering these questions.

(a) (6 points) How can the FMM be used to solve an integral equation?
(b) (6 points) What does the term “diagonal form of a translational operator” mean?
(c) (7 points) What is the most expensive step in the regular multilevel fast multipole method? How does it scale with dimensionality of the problem?
(d) (6 points) What are structured matrices? What are Toeplitz and Hankel matrices? How can they be used in the FMM?
(e) (4 points) What is the asymptotic complexity of the step that sets the data structure for the MLFMM?
(f) (6 points) Write down the regular and singular basis functions commonly used for the Laplace equation in three dimensions, i.e., for the “mother function”

$$\Phi(x, y) = \frac{1}{|x - y|}.$$ 

(g) (6 points) What is the $S|R$ translation operator? Draw a sketch showing the domains of expansion of the $S|R$ translation.
(h) (6 points) What are the problems in the use of the multi-level FMM in higher dimensions?
(i) (3 points) Other than your own class project, name two other projects and identify their “mother functions”

2. (50 points) To achieve uniform error, the expansions for the Helmholtz equation require that the number of terms retained in the expansion (truncation number) be increased proportional to the translation distance. Consider the 2-D problem for the evaluation of the sum

$$v(y_j) = \frac{1}{N} \sum_{i=1}^{N} \Phi(y_j, x_i) u_i, \quad j = 1, \ldots, N,$$

where $\Phi$ has this property. The points $\{y_j\}$ and $\{x_i\}$ are distributed uniformly in a 2-D square domain. At the finest level $L$ it is sufficient to retain $p_L$ terms to achieve a specified error $\epsilon$.

(a) Derive an expression for $p_{L-1}$ as a function of $p_L$.
(b) Derive an expression for $p_l$, the truncation number at level $l$, as a function of $p_L$.
(c) Evaluate the cost of the upward pass for the Multi-level FMM. Assume that the translation from level $l$ to $l - 1$ is performed using a matrix-vector multiply of a translation matrix of size $p_{l-1} \times p_l$ times a coefficient vector of size $p_l$.
(d) Determine the cost of the $S|R$ translations in the downward pass. Assume that the $S|R$ translation at level $l$ is performed using a matrix-vector multiply of a translation matrix of size $p_l \times p_l$ times a coefficient vector of size $p_l$.

(e) Evaluate the cost of the $R|R$ translations in the downward pass of the Multi-level FMM. Assume that the translation from level $l-1$ to $l$ is performed using a matrix-vector multiply of a translation matrix of size $p_l \times p_{l-1}$ times a coefficient vector of size $p_{l-1}$.

(f) Evaluate the total cost of the FMM.

(g) Determine the optimal clustering parameter $s$, where

$$s = \frac{N}{2^{2L}},$$

for minimum cost. (Hint. Write the expression in part (f) in terms of $s$. Then simplify the expression you obtain to keep only the leading growing and decaying terms in $s$.)