Outline

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• Matrix representation of operators
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• S|R and R|S translation operators
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Why do we need represent functions in different spaces?

- Functions should be efficiently summed up;
- Sums of functions should be compressed;
- Error bounds should be established;
- Functions should be translated and expanded over different bases;
- For computations we need discrete and finite function representations.
- Some functions measured experimentally or approximated by splines, and there is no explicit analytical representation in the whole space.
Function Representation in the Space of Coefficients

Let $F(\Omega) \subset C(\Omega)$, $\Omega \subset \mathbb{R}^d$, be a normed space of continuous functions with norm

$$
\| \Phi(y) \| = \max_{y \in \Omega} |\Phi(y)|.
$$

Let also $\{F_n(y)\}$ be a complete basis in $F(\Omega)$, so

$$
\Phi(y) = \sum_{n=0}^{\infty} A_n F_n(y), \quad y \in \Omega \subset \mathbb{R}^d, \quad \Phi(y), F_n(y) \in F(\Omega),
$$

absolutely and uniformly converges in $\Omega \subset \mathbb{R}^d$. This means that

$$
\forall \epsilon > 0, \quad \exists p(\epsilon), \quad |\Phi(y) - \Phi^p(y)| < \epsilon, \quad \forall y \in \Omega,
$$

$$
\forall \epsilon > 0, \quad \exists p(\epsilon), \quad \sum_{n=p}^{\infty} |A_n F_n(y)| < \epsilon, \quad \forall y \in \Omega,
$$

$$
\Phi^p(y) = \sum_{n=0}^{p-1} A_n F_n(y).
$$
Function Representation in the Space of Coefficients (2)

Expansion coefficients can be stacked in an infinite vector

\[ A = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \\ \vdots \end{pmatrix}. \]

Let us denote \( A(\Omega) \) a subset of \( \mathbb{R}^\infty \) which is an image of \( F(\Omega) \). For any \( A \in A(\Omega) \) we request that there exists one-to-one mapping

\[ \Phi(y) \implies A, \quad \Phi(y) \in F(\Omega), \quad A \in A(\Omega) \subset \mathbb{R}^\infty. \]
p-Truncated Vectors

\[ \forall A \in \mathbb{R}^p, \exists \Phi^p(y) = \sum_{n=0}^{p-1} A_n F_n(y) \in F^p(\Omega) \subset F(\Omega). \]

\( F^p(\Omega) \) is dense in \( F(\Omega) \):

\[ \forall \Phi(y) \in F(\Omega), \exists p, \Phi^p(y) \in F^p(\Omega), \quad \| \Phi(y) - \Phi^p(y) \| = \min_{r \in \Omega} |\Phi(y) - \Phi^p(y)| < \varepsilon. \]
Matrix Representation of Linear Operators

Let $\Omega' \subset \Omega$ and $\mathcal{F}$ is a mapping of $\mathcal{F}(\Omega)$ to $\mathcal{F}(\Omega')$. Such mapping can be considered as action of operator $\mathcal{F}$ on $\Phi$:

$$\mathcal{F}[\Phi(y)] = \widetilde{\Phi}(y), \quad \Phi(y) \in \mathcal{F}(\Omega), \quad \widetilde{\Phi}(y) \in \mathcal{F}(\Omega') \subset \mathcal{F}(\Omega)$$

Respectively, operator $\mathcal{F}$ generates operator $\mathcal{F}'$ that maps the space of expansion coefficients $\mathcal{A}(\Omega) \to \mathcal{A}(\Omega')$, which can be considered as representation of the operator $\mathcal{F}$ in the space of expansion coefficients:

$$\mathcal{F}A = \widetilde{\mathcal{A}}, \quad \mathcal{A} \in \mathcal{A}(\Omega), \quad \widetilde{\mathcal{A}} \in \mathcal{A}(\Omega') \subset \mathcal{A}(\Omega).$$

Inversely, if we introduce any transform of expansion coefficients $\mathcal{F}A = \widetilde{\mathcal{A}}$ which provides uniform convergence of function $\widetilde{\Phi(y)}$ corresponding to these coefficients in $\Omega' \subset \Omega$ then such transform can be treated as operator $\mathcal{F}$ that convert one function from $\mathcal{F}(\Omega)$ to another.
p-Truncation (Projection) Operator

\[ \text{Pr}(p)A = \tilde{A}, \quad \tilde{A} \in \Lambda^p(\Omega). \]

\[
A = \begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_{p-1} \\
A_p \\
A_{p+1} \\
\vdots
\end{pmatrix} \quad \rightarrow \quad \tilde{A} = \begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_{p-1} \\
0 \\
0 \\
\vdots
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} A
\]

In space \( F(\Omega) \):

\[ \text{Pr}(p)[\Phi(y)] = \Phi^p(y), \quad \Phi(y) \in F(\Omega), \quad \Phi^p(y) \in F^p(\Omega), \]

\[ \lim_{p \to \infty} \| \Phi(y) - \text{Pr}(p)[\Phi(y)] \| = 0. \]
Norm of p-Truncation Operator (important for error bounds)

Norm:

$$\|Pr(p)\| = \frac{\sup_{y \in \Omega} \|Pr(p)[\Phi(y)]\|}{\sup_{y \in \Omega} \|\Phi(y)\|}.$$ 

Triangle inequality:

$$\|I\| - \|I - Pr(p)\| \leq \|Pr(p)\| \leq \|I\| + \|I - Pr(p)\| = 1 + \|I - Pr(p)\|$$

so

$$\forall \epsilon > 0, \ \exists p, \ \|I - Pr(p)\| < \epsilon,$$

so

$$\forall \epsilon > 0, \ \exists p, \ 1 - \epsilon < \|Pr(p)\| < 1 + \epsilon,$$
p-Truncated Operator

Let $H : F(\Omega) \to F(\Omega)$ be an operator, that is represented by infinite matrix

$$H = \begin{pmatrix}
  h_{00} & h_{01} & \ldots & h_{0,p-1} & h_{0,p} & \ldots \\
  h_{10} & h_{11} & \ldots & h_{1,p-1} & h_{1,p} & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
  h_{p-1,0} & h_{p-1,1} & \ldots & h_{p-1,p-1} & h_{p-1,p} & \ldots \\
  h_{p0} & h_{p1} & \ldots & h_{p-1,p} & h_{pp} & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix}.$$

We call operator $H^{(p)} : F(\Omega) \to F(\Omega)$, \textit{p-truncated} if it is represented by matrix

$$H^{(p)} = \begin{pmatrix}
  h_{00} & h_{01} & \ldots & h_{0,p-1} & 0 & \ldots \\
  h_{10} & h_{11} & \ldots & h_{1,p-1} & 0 & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
  h_{p-1,0} & h_{p-1,1} & \ldots & h_{p-1,p-1} & 0 & \ldots \\
  0 & 0 & \ldots & 0 & 0 & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix}.$$
Norm of p-Truncated Operator
(important for error bounds)

**Theorem:** Let $H : F(\Omega) \to F(\Omega)$, such that $0 < \|H\| < \infty$, and $H^{(p)} : F(\Omega) \to F(\Omega)$ is the $p$-truncated operator $H$. Let also $p(\varepsilon)$ be such that $1 - \varepsilon < \|Pr(p)\| < 1 + \varepsilon$. Then

$$(1 - \varepsilon)^2 < \|Pr(p)\|^2 = \frac{\|H^{(p)}\|}{\|H\|} = \|Pr(p)\|^2 < (1 + \varepsilon)^2,$$

$$\lim_{p \to \infty} \frac{\|H^{(p)}\|}{\|H\|} = 1.$$

**Proof.**

A $p$-truncated operator can be represented in the form

$$H^{(p)} = Pr(p)HPr(p)$$

(check!)

So the norm of $H^{(p)}$ is

$$\|H^{(p)}\| = \|Pr(p)\|\|H\|\|Pr(p)\| = \|H\|\|Pr(p)\|^2.$$

End of Proof.
Translation Operator

Operator $T(t) : \mathcal{F}(\Omega) \rightarrow \mathcal{F}(\Omega')$, $\Omega' \subset \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is called translation operator corresponding to translation vector $t$, if

$$T(t)[\Phi(y)] = \Phi(y + t), \quad (y \in \Omega, \ y + t \in \Omega').$$
Example of Translation Operator

\[ \Phi(y+t) \]

The diagram illustrates the translation operator \( \Phi \) applied to \( y \) plus a shift \( t \). The function \( \Phi(y+t) \) is depicted with a dashed line, shifted to the right compared to \( \Phi(y) \), which is shown with a solid line.
R|R-reexpansion

Let \( y - x_\ast \in \Omega_r(x_\ast) \subset \mathbb{R}^d \), \( \Omega_r(x_\ast) : |y - x_\ast| < r \), and \( \{ R_n(y - x_\ast) \} \) be a regular basis in \( C(\Omega) \). Let \( y - x_\ast + t \in \Omega_r(x_\ast) \) and

\[
R_n(y - x_\ast + t) = \sum_{l=0}^{\infty} (R|R)_{ln}(t) R_l(y - x_\ast).
\]

Coefficients \( (R|R)_{ln}(t) \) are called \( R|R - reexpansion coefficients \) (regular-to-regular), and infinite matrix

\[
(R|R)(t) = \begin{pmatrix}
(R|R)_{00} & (R|R)_{01} & \cdots \\
(R|R)_{10} & (R|R)_{11} & \cdots \\
\vdots & \vdots & \ddots \\
\end{pmatrix}
\]

is called \( R|R - reexpansion matrix \).
Example of R|R-reexpansion

\[ R_m(x) = x^m, \]
\[ R_m(x + t) = (x + t)^m = x^m + \binom{m}{1} x^{m-1} t + \cdots + \binom{m}{m-1} x t^{m-1} + t^m \]
\[ = \sum_{l=0}^{m} \binom{m}{l} t^l x^{m-l} = \sum_{l=0}^{m} \binom{m}{l} t^{m-l} x^l = \sum_{l=0}^{m} \binom{m}{l} t^{m-l} R_l(x), \]

\[ (R|R)_{lm}(t) = \begin{cases} \binom{m}{l} t^{m-l}, & l \leq m, \\ 0, & l > m. \end{cases} \]
R|R-translation operator

Translation operator $T(t)$ which is represented in regular basis $\{R_x(y - x_\ast)\}$ by the $R|R - reexpansion matrix$ is called $R|R$-translation operator.

$$T(t)[\Phi(y)] = \Phi(y + t)$$

$$(R|R)(t) = T(t).$$
Why the same operator named differently?

\[ \overline{T}(t)[\Phi(y)] = \Phi(y + t) \]

The first letter shows the basis for \( \Phi(y) \)

\[ \overline{T}(t) = \begin{cases} 
(\mathcal{R}|\mathcal{R})(t) \\
(\mathcal{S}|\mathcal{S})(t) \\
(\mathcal{S}|\mathcal{R})(t) \\
(\mathcal{R}|\mathcal{S})(t) 
\end{cases} \]

The second letter shows the basis for \( \Phi(y + t) \)

Needed only to show the expansion basis (for operator representation)
Matrix representation of $R|\mathcal{R}$-translation operator

Consider

$$\Phi(y) = \sum_{n=0}^{\infty} A_n(x_*) R_n(y - x_*).$$

$$\Phi(y + t) = (R|\mathcal{R})(t) [\Phi(y)] = \sum_{n=0}^{\infty} A_n(x_*) (R|\mathcal{R})(t) [R_n(y - x_*)] = \sum_{n=0}^{\infty} A_n(x_*) R_n(y - x_* + t)$$

$$= \sum_{n=0}^{\infty} A_n(x_*) \sum_{l=0}^{\infty} (R|\mathcal{R})_{ln}(t) R_l(y - x_*).$$

$$= \sum_{l=0}^{\infty} \left[ \sum_{n=0}^{\infty} (R|\mathcal{R})_{ln}(t) A_n(x_*) \right] R_l(y - x_*).$$

$$= \sum_{l=0}^{\infty} \widetilde{A}_l(x_*, t) R_l(y - x_*),$$

$$\widetilde{A}_l(x_*, t) = \sum_{n=0}^{\infty} (R|\mathcal{R})_{ln}(t) A_n(x_*), \quad \widetilde{A}(x_*, t) = (R|\mathcal{R})(t) \widetilde{A}(x_*).$$
Reexpansion of the same function over shifted basis

Compact notation:

\[
\Phi(y) = \sum_{n=0}^{\infty} A_n(x_*) R_n(y - x_*) = A(x_*) \circ R(y - x_*)
\]

\[
\Phi(y + t) = \sum_{j=0}^{\infty} \tilde{A}_j(x_*, t) R_j(y - x_*) = \tilde{A}(x_*, t) \circ R(y - x_*)
\]

We have:

\[
\Phi(y) = \Phi((y - t) + t) = \tilde{A}(x_*, t) \circ R((y - t) - x_*)
\]

\[
= \tilde{A}(x_*, t) \circ R(y - x_* - t).
\]

Also

\[
\Phi(y) = A(x_*) \circ R(y - x_*) = A(x_* + t) \circ R(y - x_* - t),
\]

so

\[
A(x_* + t) = \tilde{A}(x_*, t) = (R|R)(t)A(x_*).
\]
R|R-reexpansion of the same function over shifted basis (2)

|y - x* - t| < r_1 = r - |t|

Since Ω_{r_1}(x_*+t) ⊂ Ω_r(t)!

Original expansion Is valid only here!
Example of power series reexpansion \[ R_m(x) = x^m. \]

\[
\Phi(y, x_i) = \sum_{m=0}^{\infty} A_m(x_{*1}, x_i) R_m(y - x_{*1}) = \sum_{m=0}^{\infty} A_m(x_{*2}, x_i) R_m(y - x_{*2}),
\]

\[
A(x_{*2}, x_i) = (R|R)(x_{*2} - x_{*1}) \cdot A(x_{*1}, x_i).
\]

\[
\begin{pmatrix}
A_0(x_{*2}, x_i) \\
A_1(x_{*2}, x_i) \\
A_2(x_{*2}, x_i) \\
\vdots
\end{pmatrix}
= 
\begin{pmatrix}
1 & \begin{pmatrix} 1 \\ 0 \end{pmatrix}(x_{*2} - x_{*1}) & \begin{pmatrix} 2 \\ 0 \end{pmatrix}(x_{*2} - x_{*1})^2 & \ldots \\
0 & 1 & \begin{pmatrix} 2 \\ 1 \end{pmatrix}(x_{*2} - x_{*1}) & \ldots \\
0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \ldots
\end{pmatrix}
\begin{pmatrix}
A_0(x_{*1}, x_i) \\
A_1(x_{*1}, x_i) \\
A_2(x_{*1}, x_i) \\
\vdots
\end{pmatrix}
\]
Example of power series reexpansion (2). Relation to Taylor series.

Let's check this for Taylor series, when expansion coefficients are

$$A_m(x_{*1}, x_{*i}) = \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_{*i})}{\partial x_{*1}^m}$$

For $A_0$ this yields Taylor series again!

Check for $A_l$

$$\Phi(x_{*2}, x_{*i}) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_{*i})}{\partial x_{*1}^m} (x_{*2} - x_{*1})^m,$$

$$\frac{1}{l!} \frac{\partial^l \Phi(x_{*2}, x_{*i})}{\partial x_{*2}^l} = \sum_{m=l}^{\infty} \binom{m}{l} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_{*i})}{\partial x_{*1}^m} (x_{*2} - x_{*1})^{m-l}$$

$$= \sum_{m=l}^{\infty} \frac{m!}{l!(m-l)!} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_{*i})}{\partial x_{*1}^m} (x_{*2} - x_{*1})^{m-l}$$

$$= \frac{1}{l!} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{k+l} \Phi(x_{*1}, x_{*i})}{\partial x_{*1}^{k+l}} (x_{*2} - x_{*1})^k,$$

$$\frac{\partial^l \Phi(x_{*2}, x_{*i})}{\partial x_{*2}^l} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{k+l} \Phi(x_{*1}, x_{*i})}{\partial x_{*1}^{k+l}} (x_{*2} - x_{*1})^k.$$
S|S-reexpansion

Let \( y - x_* \in \Omega_r(x_*) \subset \mathbb{R}^d \), \( \Omega_r(x_*) : |y - x_*| > r \), and \( \{S_n(y - x_*)\} \) be a singular basis in \( C(\Omega) \). Let \( y - x_* + t \in \Omega_r(x_*) \) and

\[
S_n(y - x_* + t) = \sum_{l=0}^{\infty} (S|S)_{ln}(t)S_l(y - x_*).
\]

Coefficients \( (S|S)_{ln}(t) \) are called \( S|S - \text{reexpansion coefficients} \) (singular-to-singular), and infinite matrix

\[
(S|S)(t) = \begin{pmatrix}
(S|S)_{00} & (S|S)_{01} & \cdots \\
(S|S)_{10} & (S|S)_{11} & \cdots \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\]

is called \( S|S - \text{reexpansion matrix} \).
S|S-translation operator

Translation operator $T(t)$ which is represented in singular basis $\{S_n(y - x_*)\}$ by the $S|S - reexpansion matrix$ is called $S|S$-translation operator.

$$T(t)[\Phi(y)] = \Phi(y + t)$$

$$(S|S)(t) = T(t).$$
S|S and R|R-translation operators are very similar,

(actually, this is just two representations of the same translation operator in different domains and bases)

\[ \Phi(y) = B(x_*) \circ S(y - x_*), \]
\[ \Phi(y + t) = \tilde{B}(x_*, t) \circ S(y - x_*). \]
\[ \Phi(y) = \tilde{B}(x_*, t) \circ S(y - x_* - t). \]
\[ \tilde{B}(x_*, t) = (S|S)(t)B(x_*) = B(x_* + t). \]
But picture is different…

Original expansion
Is valid only here!

\[ |y - x_\ast - t| > r_1 = r + |t| \]

Since
\[ \Omega_{r_1}(x_\ast + t) \subset \Omega_r(t) ! \]

Also
\[ |x_i - x_\ast| < r \]

singular point!
S|R-reexpansion

Let \( y - x_* \in \Omega_r(x_*) \subset \mathbb{R}^d \), \( \Omega_r(x_*) : |y - x_*| < r \), and \( \{R_n(y - x_*)\} \) be a regular basis in \( C(\Omega_r(x_*)) \). Let also \( \Omega_{r1}(x_* - t) : |y - x_* + t| > R > r \), and \( \{S_n(y - x_* + t)\} \) be a singular basis in \( C(\Omega_r(x_*)) \), then

\[
S_n(y - x_* + t) = \sum_{j=0}^{\infty} (S|R)_{jn}(t)R_j(y - x_*).
\]

Coefficients \( (S|R)_{jn}(t) \) are called \( S|R - reexpansion coefficients \) (singular-to-regular), and infinite matrix

\[
(S|R)(t) = \begin{pmatrix}
(S|R)_{00} & (S|R)_{01} & \cdots \\
(S|R)_{10} & (S|R)_{11} & \cdots \\
\cdots & \cdots & \cdots
\end{pmatrix}
\]

is called \( S|R - reexpansion matrix \).
Does R|S reexpansion exist?

- Theoretically yes (in some cases, e.g. analytical continuation);
- In practice, since the domain of S-expansion is larger than the domain of R-expansion, this either not useful (due to error bounds), or can be avoided in algorithms;
- We will not use R|S-reexpansions in the FMM algorithms.
S|R-translation operator

Translation operator $\mathcal{T}(t)$ which is represented in singular basis by the $\mathcal{S}\mathcal{R}$-reexpansion matrix is called $\mathcal{S}\mathcal{R}$-translation operator if the basis of expansion is changed with the translation operation from singular $\{S_n(y - x_*)\}$ to regular $\{R_n(y - x_* + t)\}$

$$\mathcal{T}(t)[\Phi(y)] = \Phi(y + t)$$

$$(\mathcal{S}\mathcal{R})(t) = \mathcal{T}(t).$$
S|R-operator has almost the same properties as S|S and R|R

(t cannot be zero)

\[ \Phi(y) = B(x_*) \circ S(y - x_*) , \]
\[ \Phi(y + t) = \tilde{A}(x_*, t) \circ R(y - x_*) \]
\[ \Phi(y) = \tilde{A}(x_*, t) \circ R(y - x_* - t). \]

\[ \tilde{A}(x_*, t) = (S|R)(t)B(x_*). \]
Since
\[ \Omega_{r_1}(x_* + t) \subset \Omega_r(t) \]

Also
\[ |x_i - x_*| < r \]
singular point!
Properties of the translation operator

\[ T(t)[\Phi(y)] = \Phi(y+t) \]

- \( T(0) = T \) (identity operator). Proof:
  \[ T(0)[\Phi(y)] = \Phi(y). \]

- \( T(t_1 + t_2) = T(t_1) \circ T(t_2) = T(t_2) \circ T(t_1) \). Proof:
  \[ T(t_1) \circ T(t_2)[\Phi(y)] = \Phi(y + t_2 + t_1) = T(t_2 + t_1)[\Phi(y)] = T(t_1 + t_2)[\Phi(y)]. \]

- (corollary 1): \( T^{-1}(t) = T(-t) \). Proof:
  \[ T = T(0) = T(t-t) = T(t) \circ T(-t). \]

- (corollary 2): \( T^n(t) = T(nt) \). Proof (use induction):
  \[ T(nt) = T((n-1)t) \circ T(t) = T^{n-1}(t) \circ T(t) = T^n(t). \]
Spectrum of the translation operator

\[ \mathcal{T}(t)[\Psi(y)] = \lambda \Psi(y), \quad y \in \mathbb{R}^d. \]

Any function of type

\[ \forall a \in \mathbb{R}^d, \quad \Psi(y) = e^{ay}, \quad \lambda = e^{at}. \]

Check:

\[ \mathcal{T}(t)[\Psi(y)] = \Psi(y + t) = e^{a(y + t)} = e^{at}e^{ay} = \lambda \Psi(y). \]

Relation to differential operator:

\[ \frac{d\Phi(y)}{ds} = \lim_{|t| \to 0} \frac{\Phi(y + t) - \Phi(y)}{|t|} = \lim_{|t| \to 0} \frac{\mathcal{T}(t)[\Phi(y)] - \Phi(y)}{|t|} = \lim_{|t| \to 0} \frac{\mathcal{T}(t) - \mathcal{T}}{|t|} \Phi(y), \quad s = \frac{t}{|t|}. \]

derivative in direction s