The Multilevel Fast Multipole Method

Ramani Duraiswami
Nail Gumerov
Outline

• Review
• Vector analysis (Divergence & Gradient of potential)
• 3-D Cartesian coordinates & Spherical coordinates
• Laplace’s equation and Helmholtz’ equation
• Green's function & Green's theorem
• Boundary element method
• FMM
Gauss Divergence theorem

- The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

\[ \int_{\Omega} \nabla \cdot A \, dV = \int_{S} A \cdot n \, dS \]

Proof follows from the definition of divergence.

- In practice we can write

\[ \int_{\Omega} \nabla \text{anything} \, dV = \int_{S} \text{anything} \, n \, dS \]
Integral Definitions of \( \text{div}, \text{grad} \) and \( \text{curl} \)

\[
\nabla \phi \equiv \lim_{\delta \tau \to 0} \frac{1}{\delta \tau} \int_{\Delta S} \phi \mathbf{n} dS
\]

\[
\nabla \cdot \mathbf{D} \equiv \lim_{\delta \tau \to 0} \frac{1}{\delta \tau} \int_{\Delta S} \mathbf{D} \cdot \mathbf{n} dS
\]

\[
\nabla \times \mathbf{D} \equiv -\lim_{\delta \tau \to 0} \frac{1}{\delta \tau} \int_{\Delta S} \mathbf{D} \times \mathbf{n} dS
\]

\( \mathbf{D} = \mathbf{D}(\mathbf{r}), \phi = \phi(\mathbf{r}) \)

Elemental volume \( \delta \tau \) with surface \( \Delta S \)
Green’s formula

Green’s first theorem

$$\int_{\Omega} (\psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi) dV = \int_{S} \psi \frac{\partial \phi}{\partial n} dS$$

$$\int_{\Omega} (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) dV = \int_{S} \phi \frac{\partial \psi}{\partial n} dS$$

Green’s second theorem (subtracting the above two)

$$\int_{\Omega} (\psi \nabla^2 \phi - \phi \nabla^2 \psi) dV = \int_{S} \left( \psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) dS$$
Laplace’s equation

Let $G$ satisfy

$$\nabla^2 G (x) = \delta (x)$$

Solution is

$$G (x) = -\frac{1}{4\pi r}$$

More generally

$$G (x, y) = -\frac{1}{4\pi |x - y|}$$
Helmholtz equation

Let $G$ satisfy

$$\nabla^2 G(x) + k^2 G = \delta(x)$$

Solution is

$$G(x) = -\frac{\exp(ikr)}{4\pi r}$$

More generally

$$G(x, y) = -\frac{\exp(ik|x - y|)}{4\pi |x - y|}$$
Green’s formula

\[ \psi(x) = \int_{S_y} \left( \psi(y) \frac{\partial G(x, y)}{\partial n_y} - G(x, y) \frac{\partial \psi}{\partial n_y}(y) \right) dS_y \]

- Discretize surface \( S \) into triangles
- Discretize local function in terms of local isoparametric shape functions, i.e., as:

\[ \phi(x) = \sum_{i=1}^{N} \phi_i N_i(x), \quad q(x) = \sum_{i=1}^{N} q_i N_i(x), \]

where usually:

\[ N_i(x) = \begin{cases} 1, & x \in S_i \\ 0, & x \notin S_i \end{cases} \quad \text{for constant elements.} \]
Green’s formula

• Recall that the impulse-response is sufficient to characterize a linear system
• Solution to arbitrary forcing constructed via convolution
• For a linear boundary value problem we can likewise use the solution to a delta-function forcing to solve it.
• Fluid flow, steady-state heat transfer, gravitational potential, etc. can be expressed in terms of Laplace’s equation

\[ \nabla^2 P = 0 \quad \nabla^2 P + k^2 P = 0 \]

• Solution to delta function forcing, without boundaries, is called free-space Green’s function

\[
P(x) = -\int_s \left[ \frac{\partial P(y)}{\partial n}(y)G(x, y) - P(y)\frac{\partial G(x, y)}{\partial n} \right] dS(y), \quad x \notin \Omega,
\]
Boundary Element Methods

With this discretization we can write Green identity in the form

$$\frac{1}{2} \phi_j = \sum_{i=1}^{N} q_i \int_{S_i} G(x - y_j) \, dS(x) - \sum_{i=1}^{N} \phi_i \int_{S_i} \frac{\partial G(x - y_j)}{\partial n_x} \, dS(x),$$

- Boundary conditions provide value of $\phi_j$ or $q_j$
- Becomes a linear system to solve for the other
Accelerate via FMM

\[ \Phi_i(y) = \int_{S_i} G(x - y)dS(x), \]
\[ Q_i(y) = \int_{S_i} \frac{\partial G(x - y)}{\partial n_x}dS(x), \]

Using the expansion of the Green function

\[ G(r - r_{*1}) = G(r_{*1} - r) = ik \sum_{n=0}^{\infty} \sum_{m} R_{n}^{-m}(r_{*1} - r_{*2})S_{n}^{m}(r - r_{*2}), \quad |r - r_{*2}| > |r|. \]
Therefore, for such a \( \mathbf{y} \) we have

\[
\Phi_i(\mathbf{y}) = \int_{S_i} G(\mathbf{x} - \mathbf{y})dS(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} S_n^m(\mathbf{y} - \mathbf{x}^{(n,L)})i^k \int_{S_i} R_n^{-m}(\mathbf{x} - \mathbf{x}^{(n,L)})dS(\mathbf{x}).
\]

Comparing with Eq. (ref: e1), we can determine the expansion coefficients \( A_n^{(i)m} \):

\[
A_n^{(i)m} = ik \int_{S_i} R_n^{-m}(\mathbf{x} - \mathbf{x}^{(n,L)})dS(\mathbf{x}).
\]

Similarly, we consider \( Q_i(\mathbf{y}) \). Here we note that since the triangle is flat its normal, \( \mathbf{n}_i \), does not change. Therefore,

\[
Q_i(\mathbf{y}) = \int_{S_i} \frac{\partial G(\mathbf{x} - \mathbf{y})}{\partial n_x}dS(\mathbf{x}) = \mathbf{n}_i \cdot \int_{S_i} \nabla_x G(\mathbf{x} - \mathbf{y})dS(\mathbf{x})
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} S_n^m(\mathbf{y} - \mathbf{x}^{(n,L)})i^k \mathbf{n}_i \cdot \int_{S_i} \nabla_x R_n^{-m}(\mathbf{x} - \mathbf{x}^{(n,L)})dS(\mathbf{x}).
\]
\[ Q_i(y) = \int_{S_i} \frac{\partial G(x - y)}{\partial n_x} dS(x) = n_i \cdot \int_{S_i} \nabla_x G(x - y) dS(x) \]
\[ = -(n_i \cdot \nabla_y) \int_{S_i} G(x - y) dS(x) = -(n_i \cdot \nabla_y) \Phi_i(y). \]