Lecture 4
Example

$$\Phi(y, x_i) = e^{y \cdot x_i} = \sum_{m=0}^{\infty} \frac{1}{m!} [(y - x_*) \cdot \nabla_{x_*}]^m \Phi(x_*, x_i),$$

Fix ($y - x_*$):

$$\Phi(x_*, x_i) = e^{x_* \cdot x_i},$$

$$\nabla_{x_*} \Phi(x_*, x_i) = x_i e^{x_* \cdot x_i} = x_i \Phi(x_*, x_i),$$

$$[(y - x_*) \cdot \nabla_{x_*}] \Phi(x_*, x_i) = [(y - x_*) \cdot x_i] \Phi(x_*, x_i),$$

$$[(y - x_*) \cdot \nabla_{x_*}]^m \Phi(x_*, x_i) = [(y - x_*) \cdot x_i]^m \Phi(x_*, x_i),$$

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} \frac{1}{m!} [(y - x_*) \cdot x_i]^m \Phi(x_*, x_i) = e^{x_* \cdot x_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(y - x_*) \cdot x_i]^m.$$

Check: $$e^{y \cdot x_i} = e^{x_* \cdot x_i} e^{(y-x_*) \cdot x_i} = e^{x_* \cdot x_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(y - x_*) \cdot x_i]^m.$$
Is That a Factorization?

\[ e^{y \cdot x_i} = e^{x_\star \cdot x_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(y - x_\star) \cdot x_i]^m \]
Scalar Product in d-Dimensional Space

Definition of scalar product:

\[ \mathbf{a} = (a_1, \ldots, a_d), \quad \mathbf{b} = (b_1, \ldots, b_d), \]

\[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \ldots + a_d b_d = \sum_{k=1}^{d} a_k b_k. \]

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta, \]

\[ |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}. \]

What if \( a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{C} \) ?

Definition:

\[ \mathbf{a} \cdot \mathbf{b} = \overline{a_1} b_1 + \ldots + \overline{a_d} b_d = \sum_{k=1}^{d} \overline{a_k} b_k. \]
Properties of Scalar Product

Commutativity:

\[ a \cdot b = b \cdot a \]

Scaling:

\[ (\lambda a) \cdot b = a \cdot (\lambda b) = \lambda (a \cdot b), \quad \lambda \in \mathbb{R} \]

Distributivity:

\[ (a + b) \cdot c = a \cdot c + b \cdot c \]
Factorization of Scalar Product Powers

\[(a \cdot b)^n = \left( \sum_{k=1}^{d} a_k b_k \right)^n = \sum_{k_1=1}^{d} a_{k_1} b_{k_1} \sum_{k_2=1}^{d} a_{k_2} b_{k_2} \cdots \sum_{k_n=1}^{d} a_{k_n} b_{k_n} \]

\[= \sum_{k_1=1}^{d} \sum_{k_2=1}^{d} \cdots \sum_{k_n=1}^{d} a_{k_1} a_{k_2} \cdots a_{k_n} b_{k_1} b_{k_2} \cdots b_{k_n} \]

\[= [a \otimes a \otimes \cdots \otimes a] \cdot [b \otimes b \otimes \cdots \otimes b] = a^n \cdot b^n \]

\[a^n \cdot b^n = (a \cdot b)^n = (b \cdot a)^n = b^n \cdot a^n.\]

\[e^{y \cdot x_i} = e^{x_i \cdot x_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(y - x_\ast) \cdot x_i]^m = e^{x_i \cdot x_i} \sum_{m=0}^{\infty} \frac{1}{m!} x_i^m \cdot (y - x_\ast)^m.\]
Is That Factorization?

1) Truncation:

\[
\Phi(y, x_i) = e^{y^*x_i} = e^{x_i^*x_i} \left[ \sum_{m=0}^{p-1} \frac{1}{m!} x_i^m \cdot (y - x_*)^m + \text{Residual}_p \right]
\]

2) Fast summation:

\[
v_j = \sum_{i=1}^{N} u_i \Phi(y_j, x_i) = \sum_{i=1}^{N} u_i e^{x_i^*x_i} \left[ \sum_{m=0}^{p-1} \frac{1}{m!} x_i^m \cdot (y_j - x_*)^m + \text{Residual}_p \right]
\]

\[
= \sum_{i=1}^{N} u_i e^{x_i^*x_i} \sum_{m=0}^{p-1} \frac{1}{m!} x_i^m \cdot (y_j - x_*)^m + N \max_i (u_i e^{x_i^*x_i}) \text{Residual}_p
\]

\[
= \sum_{m=0}^{p-1} \frac{1}{m!} \left( \sum_{i=1}^{N} u_i e^{x_i^*x_i} x_i^m \right) \cdot (y_j - x_*)^m + \text{Residual}
\]

\[
= \sum_{m=0}^{p-1} c_m \cdot (y_j - x_*)^m + \text{Residual}, \quad c_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{x_i^*x_i} x_i^m.
\]

Yes! It is!
Complexity of Product

\[ a = (a_1, a_2), \]
\[ a^2 = (a_1(a_1, a_2), a_2(a_1, a_2)) = (a_2, a_1a_2, a_2a_1, a_2^2), \]
\[ a^3 = (a_3^2(a_1, a_2), a_1a_2(a_1, a_2), a_2a_1(a_1, a_2), a_2^3(a_1, a_2)) \]
\[ = (a_1^3, a_2^2a_1, a_1a_2^2a_1, a_1^2a_2a_1, a_2a_1a_2, a_2^2a_1a_2, a_2^3a_1, a_2^3), ... \]

The length of \( a^n \) is \( 2^n! \)  

**This is not factorial!**

In \( d \) dimensions the length of \( a^n \) is even \( d^n \)

What to do in practical problems?
Use Compression!

Compression operator:

\[ A^n = \text{Compress}(a^n) \]

Required Property:

\[ a^n \cdot b^n = \text{Compress}(a^n) \cdot \text{Compress}(b^n). \]

Consider \( R^2 \):

\[ a^n \cdot b^n = (a \cdot b)^n = (a_1 b_1 + a_2 b_2)^n \]

\[ = a^n b^n + \binom{n}{1} a_1^{n-1} b_1^{n-1} a_2 b_2 + \binom{n}{2} a_1^{n-2} b_1^{n-2} a_2^2 b_2^2 + \ldots + a_2^n b_2^n \]

The length is only \((n + 1)\), not \(2^n\)

Let us define:

\[ A^n = \text{Compress}(a^n) = \left( a_1^n, \sqrt[n]{\binom{n}{1} a_1^{n-1} a_2}, \sqrt[n]{\binom{n}{2} a_1^{n-2} a_2^2}, \ldots, a_2^n \right) \]

\[ B^n = \text{Compress}(b^n) = \left( b_1^n, \sqrt[n]{\binom{n}{1} b_1^{n-1} b_2}, \sqrt[n]{\binom{n}{2} b_1^{n-2} b_2^2}, \ldots, b_2^n \right) \]
Example of Fast Computation

\[ v_j = \sum_{i=1}^{N} u_i \Phi\left( y_j, x_i \right) = \sum_{m=0}^{p-1} c_m \cdot \left( y_j - x_* \right)^m + \text{Residual}, \quad c_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{x_* \cdot x_i} x_i^m. \]

Equivalent to:

\[ v_j = \sum_{m=0}^{p-1} C_m \cdot \text{Compress}\left( \left( y_j - x_* \right)^m \right) + \text{Residual}, \quad C_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{x_* \cdot x_i} \text{Compress}(x_i^m). \]

Number of multiplications (complexity) to obtain \( v_j \):

\[ \text{Complexity} = 1 + 2 + \ldots + p = \frac{p(p + 1)}{2}. \]
Compression Can be Performed for any Dimensionality (Example for 3D):

\[
a^n \cdot b^n = (a \cdot b)^n = (a_1 b_1 + a_2 b_2 + a_3 b_3)^n
\]

\[
= \left[ (a_1 b_1 + a_2 b_2) + a_3 b_3 \right]^n = \sum_{m=0}^{n} \binom{n}{m} (a_1 b_1 + a_2 b_2)^{n-m} a_3^m b_3^m
\]

\[
= \sum_{m=0}^{n} \sum_{l=0}^{n-m} \binom{n}{m} \binom{n-m}{l} a_1^{n-m-l} b_1^{n-m-l} a_2^l b_2^l a_3^m b_3^m
\]

\[
= a_1^n b_1^n + \binom{n}{1} a_1^{n-1} b_1^{n-1} a_2 b_2 + \binom{n}{2} a_1^{n-2} b_1^{n-2} a_2^2 b_2^2 + \ldots + a_2^n b_2^n
\]

\[
+ \binom{n}{1} a_1^{n-1} b_1^{n-1} a_3 b_3 + \binom{n}{1} \binom{n-1}{1} a_1^{n-2} b_1^{n-2} a_2 b_2 a_3 b_3 + \ldots + a_2^n b_2^n,
\]

\[
\text{Compress}(a^n) = \begin{pmatrix}
a_1^n, & \binom{n}{1} a_1^{n-1} a_2, & \binom{n}{2} a_1^{n-2} a_2^2, & \ldots, & a_2^n, & \binom{n}{1} a_1^{n-1} a_3, & \ldots, & a_3^n
\end{pmatrix}
\]

The length of \(a^n\) is \((n+1)+n+\ldots+1= (n+1)(n+2)/2\)
Compression Can be Performed for any Dimensionality (General Case):

\[
(a_1 + a_2 + \ldots + a_d)^n = \sum_{n_1 + \ldots + n_d = n} (n, n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d}.
\]

\[
(n, n_1, n_2, \ldots, n_d) = \frac{n!}{n_1!n_2!\ldots n_d!}.
\]

\[
\text{Compress}(a^n) = \left( a_1^n, \sqrt{(n, n-1, 1, 0, \ldots, 0)} a_1^{n-1} a_2, \ldots, \sqrt{(n, n_1, n_2, \ldots, n_d)} a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d}, \ldots, a_d^n \right)
\]

So we have

\[
a^n \cdot b^n = \text{Compress}(a^n) \cdot \text{Compress}(b^n)
\]

\[
= \sum_{n_1 + \ldots + n_d = n} (n, n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d} b_1^{n_1} b_2^{n_2} \ldots b_d^{n_d}
\]

\[
= (a_1 b_1 + a_2 b_2 + \ldots + a_d b_d)^n = (a \cdot b)^n.
\]
What are multinomial coefficients?

(n ; n₁,n₂,…,n_d) is the number of ways of putting n different objects into d different boxes with n_k in the k-th box

\[ n_1 + n_2 + \ldots + n_d = n \]
The length of the compressed vector

\[ d = 1 : \quad 1, \]
\[ d = 2 : \quad n + 1, \]
\[ d = 3 : \quad \frac{1}{2}(n + 1)(n + 2), \]
\[ ... \]

**Theorem:** If \( a \in \mathbb{R}^d \), then the length of compressed vector \( \text{Compress}(a^n) \), is

\[
\begin{pmatrix} n + d - 1 \\ n \end{pmatrix} = \frac{(n + 1)\ldots(n + d - 1)}{(d - 1)!}.
\]

**Proof:** We have a basis for induction (see above). Let this holds for \( d \) dimensions. Consider \( d + 1 \) dimensions:

\[
((a_1 + \ldots + a_d) + a_{d+1})^n = \sum_{m=0}^{n} \binom{n}{m} (a_1 + \ldots + a_d)^m a_{d+1}^{n-m}
\]

The number of terms is then

\[
\sum_{m=0}^{n} \binom{m + d - 1}{m} = \binom{d - 1}{0} + \binom{d}{1} + \ldots + \binom{n + d - 1}{n} = \binom{n + d}{n}
\]

This proves the theorem.
Example of Fast Computation

\[ v_j = \sum_{i=1}^{N} u_i \Phi(y_j, x_i) = \sum_{m=0}^{p-1} c_m \cdot (y_j - x)_m + \text{Residual,} \quad c_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{x \cdot x_i} x_i^m. \]

Equivalent to:

\[ v_j = \sum_{m=0}^{p-1} C_m \cdot \text{Compress}((y_j - x)_m) + \text{Residual,} \quad C_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{x \cdot x_i} \text{Compress}(x_i^m). \]

Number of multiplications (complexity) to obtain \( v_j \): (in 2D case!)

\[ \text{Complexity} = 1 + 2 + \ldots + p = \frac{p(p+1)}{2}. \]

\[ C_0 = \sum_{i=1}^{N} u_i e^{x \cdot x_i}, \]

\[ C_1 = (C_{11}, C_{12}) = \sum_{i=1}^{N} u_i e^{x \cdot x_i} (x_{i1}, x_{i2}), \]

\[ C_2 = (C_{21}, C_{22}, C_{23}) = \sum_{i=1}^{N} u_i e^{x \cdot x_i} \left(x_{i1}^2, \sqrt{2} x_{i1} x_{i2}, x_{i2}^2 \right), \]
• \( C(p+d-1,d) \) is asymptotically \( d^p \) for large \( d \) and fixed \( p \)

\[
\frac{(p+d-1)!}{(p-1!)d!}
\]

Recall Stirling’s formula

\[
n! \sim \sqrt{2\pi} \ n^{(n + 1/2)} \ e^{-n}
\]
Complexity of Fast Summation

Let \( \circ \) be a scalar product of vectors \( A_i \) and \( F_j \) of length \( P(p) \) (\( p \) is the truncation number). Complexity of summation over \( i \) is then \( O(PN) \).

Complexity of scalar product operation is \( P \).

Complexity of \( M \) scalar product operations is \( O(PM) \) (for \( j = 1, ..., M \)).

Total complexity is \( O(PM + PN) \).

Fast Method is more efficient than direct only if \( O(PM + PN) < O(MN) \), so we should have

\[
P(p) \ll \min(M, N)
\]
General Forms of Factorization for Fast Summation (1)

\[ v_j = \sum_{i=1}^{N} u_i \Phi(y_j, x_i), \quad j = 1, ..., M. \]

\[ \Phi(y_j, x_i) = \sum_{m=0}^{p} a_m(x_i, x_*) f_m(y_j - x_*) + \text{Error}(p, x_i, x_*, y_j) \]

\[ = a(x_i, x_*) \cdot f(y_j - x_*) + \text{Error}. \]

How about vectors of length \( p \)

\[ v_j = \sum_{i=1}^{N} u_i e^{-\lambda_i |x_i - y_j|^2} \]

Some parameter depending on \( i \)

More general to have

\[ v_j = \sum_{i=1}^{N} u_i \Phi_i(y_j) \quad \text{or} \quad v(y) = \sum_{i=1}^{N} u_i \Phi_i(y). \]
General Forms of Factorization for Fast Summation (2)

The potential can be factorized as

$$\Phi_i(y) = A_i(x_*) \circ F(y - x_*)$$

Generalized product $\circ$ can be scalar product, contraction, etc. $A_i$ and $F$ can be real or complex vectors, tensors, etc. in $p$-dimensional space.

Requirements to the product (distributivity with respect to addition)

$$(\alpha A_i + \beta A_j) \circ F = \alpha A_i \circ F + \beta A_j \circ F.$$ 

In this case

$$v(y) = \sum_{i=1}^{N} u_i \Phi_i(y) = \sum_{i=1}^{N} u_i A_i(x_*) \circ F(y - x_*) = A(x_*) \circ F(y - x_*)$$

$$A(x_*) = \sum_{i=1}^{N} u_i A_i(x_*)$$

We do not need commutativity of $\circ$ (i.e. we do not request $A_i \circ F = F \circ A_i)(!)$.
Actually, we even do need continuous variable $y$.
The problem is to represent all matrix elements in the form

$$\Phi_{ji} = A_i \circ F_j$$

then

$$v_j = \sum_{i=1}^{N} u_i \Phi_{ji} = \sum_{i=1}^{N} u_i (A_i \circ F_j) = \left( \sum_{i=1}^{N} u_i A_i \right) \circ F_j.$$
Complexity of Fast Summation

Let $\circ$ be a scalar product of vectors $A_i$ and $F_j$ of length $P(p)$ ($p$ is the truncation number). Complexity of summation over $i$ is then $O(PN)$. Complexity of scalar product operation is $P$. Complexity of $M$ scalar product operations is $O(PM)$ (for $j = 1, \ldots, M$). Total complexity is $O(PM + PN)$. Fast Method is more efficient than direct only if $O(PM + PN) < O(MN)$, so we should have

$$P(p) \ll \min(M, N)$$
Outline

• Far Field Expansions (or S-expansions)
  – Regular Potential (Convergent Series);
  – Regular Potential (Asymptotic Series);
  – Singular Potential;

• Asymptotic Series

• Approaches for Selection of the Basis Functions
Far Field Expansions
(S-expansions)

Let \( x_* \in \mathbb{R}^d \).

We call expansion

\[
\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*)
\]

far field expansion (or S-expansion) outside a sphere

\[ |y - x_*| > R_*, \]

if the series converges for \( \forall y, |y - x_*| > R_* \).
Far Field Expansion of a Regular Potential

...sometimes like this:

\[ |y - x_*| > R_* > |x_i - x_*| \]

Can be like this:

\[ |x_i - x_*| > |y - x_*| > R_* \]

\[ |y - x_*| > R_* > |x_i - x_*| \]
Local Expansion of a Regular Potential
Can be Far Field Expansion Also
(Repeat Example from Lecture 3 )

Valid for any \( r_* < \infty \), and \( x_i \).

\[
\Phi(y, x_i) = e^{-(y-x_i)^2} = \sum_{m=0}^{\infty} a_{m2}(x_i, x_*) S_{m2}(y-x_*).
\]

We have

\[
e^{-(y-x_i)^2} = e^{-(y-x_*-(x_i-x_*)^2)} = e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} e^{2(x_i-x_*)(y-x_*)}
\]

\[
= e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} \sum_{m=0}^{\infty} \frac{2^m(x_i-x_*)^m(y-x_*)^m}{m!}.
\]

Choose

\[
a_{m2}(x_i, x_*) = e^{-(x_i-x_*)^2}(x_i-x_*)^m, \quad m = 0, 1, ..., \]

\[
S_{m2}(y-x_*) = e^{-(y-x_*)^2} \frac{2^m}{m!}(y-x_*)^m, \quad m = 0, 1, ...
\]
Asymptotic Series

\[ f(x, \epsilon) = f_0(x)\varphi_0(\epsilon) + f_1(x)\varphi_1(\epsilon) + f_2(x)\varphi_2(\epsilon) + \ldots = \sum_{n=0}^{\infty} f_n(x)\varphi_n(\epsilon) \]

\[ \lim_{\epsilon \to 0} \frac{\varphi_n(\epsilon)}{\varphi_{n+1}(\epsilon)} = 0. \]

The asymptotic expansion is \textit{uniform} in domain \( x \in \Omega \) if

\[ \forall x \in \Omega, \quad \left| f(x, \epsilon) - \sum_{n=0}^{p-1} f_n(x)\varphi_n(\epsilon) \right| = O(\varphi_p(\epsilon)). \]

Otherwise the asymptotic expansion is not uniform.
Examples of Uniform and Non-Uniform Expansions

Example of uniform expansion:

\[ f(x, \varepsilon) = \frac{1}{x + \varepsilon}, \quad x > 10 \]

\[ f(x, \varepsilon) = \frac{1}{x} (1 + \frac{\varepsilon}{x})^{-1} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^n}{x^n} \]

Example of non-uniform expansion:

\[ f(x, \varepsilon) = e^{\varepsilon x}, \quad x \in \mathbb{R}^1 \]

\[ e^{\varepsilon x} = \sum_{n=0}^{\infty} \frac{\varepsilon^n x^n}{n!}. \]

Prove that! (Hint: consider \( x \gg \varepsilon^{-1} \).)
Example of Far Field Expansion of a Regular Function (Using Asymptotic Series)

\[
\Phi(y, x_i) = \frac{1}{1 + (y - x_i)^2} = \frac{1}{1 + [y - x_\ast - (x_i - x_\ast)]^2} = \frac{1}{(y - x_\ast)^2} \frac{(y - x_\ast)^2}{1 + [y - x_\ast - (x_i - x_\ast)]^2}.
\]

Let

\[
\epsilon = \frac{1}{y - x_\ast}
\]

\[
\Phi(\epsilon, x_i - x_\ast) = \epsilon^2 \frac{1}{1 + \left[\frac{1}{\epsilon} - (x_i - x_\ast)\right]^2} = \epsilon^2 \frac{1}{\epsilon^2 + (1 - \epsilon x)^2} = \epsilon^2 f(x, \epsilon), \quad x = x_i - x_\ast
\]

\[
f(x, \epsilon) = \frac{1}{\epsilon^2 + (1 - \epsilon x)^2} = \sum_{n=0}^{\infty} f_n(x) \epsilon^n
\]

\[
f_n(x) = \frac{1}{n!} \left. \frac{\partial^n f(x, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0}
\]
Example of Far Field Expansion of a Regular Function (continuation)

\[ f_0(x) = 1, \]
\[ f_1(x) = \frac{\partial f(x, \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon = 0} = 2x, \]
\[ f_2(x) = \frac{1}{2!} \frac{\partial^2 f(x, \varepsilon)}{\partial \varepsilon^2} \bigg|_{\varepsilon = 0} = 3x^2 - 1, \]

\[ \Phi(y, x_i) = \frac{1}{(y - x_*)^2} \sum_{n=0}^{\infty} f_n(x_i - x_*) \frac{1}{(y - x_*)^n} \]

\[ y \geq 100, \quad x_i = 1, \quad x_* = 0, \]
\[ \varepsilon = 10^{-2}, \quad x = 1 \]

\[ \left| \Phi(y, x_i) - \frac{1}{(y - x_*)^2} \left[ 1 + \frac{2(x_i - x_*)}{(y - x_*)} \right] \right| \leq \varepsilon^4 (3x^2 - 1) = 2 \cdot 10^{-8}. \]
Far Field Expansion of a Singular Potential

...sometimes like this:

...sometimes like this:

Can be like this:

\[ |y - x_*| > R_* > |x_i - x_*| \]

\[ |x_i - x_*| > |y - x_*| > R_* \]

\[ |y - x_*| > R_* \geq |x_i - x_*| \]

This case only!
Example For S-expansion of Singular Potential

\[
\Phi(y, x_i) = \frac{1}{y - x_i}.
\]

\[
\frac{1}{y - x_i} = \frac{1}{y - x_* - (x_i - x_*)} = \frac{1}{(y - x_*)[1 - \frac{x_i - x_*}{y - x_*}]} = \frac{1}{(y - x_*)} \left[1 - \frac{x_i - x_*}{y - x_*}\right]^{-1}.
\]

\[
\left[1 - \frac{x_i - x_*}{y - x_*}\right]^{-1} = \sum_{m=0}^{\infty} \frac{(x_i - x_*)^m}{(y - x_*)^m}, \quad |y - x_*| > |x_i - x_*|.
\]

\[
\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*),
\]

\[
b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \ldots,
\]

\[
S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \ldots
\]
Let us compare with the R-expansion of the same function

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y-x_*), \]

\[ a_m(x_i, x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, \ldots, \]

\[ R_m(y-x_*) = (y - x_*)^m, \quad m = 0, 1, \ldots \]

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y-x_*), \]

\[ b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \ldots, \]

\[ S_m(y-x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \ldots \]

Singular Point is located at the Boundary of regions for the R- and S-expansions!
What Do We Need For Real FMM (that provides spatial grouping)

We need S-expansion for $|y - x_\ast| > R_\ast > |x_i - x_\ast|$
We need R-expansion for $|y - x_\ast| < r_\ast < |x_i - x_\ast|$
Basis Functions

• Power series are great, but do they provide the best approximation? (sometimes yes!)

• Other approaches to factorization:
  – Asymptotic Series (Can be divergent!);
  – Orthogonal Bases in $L_2$;
  – Eigen Functions of Differential Operators;
  – Functions Generated by Differentiation or Other Linear Operators.

• Some of this approaches will be considered in this course.