Outline

• Far Field Expansions (or S-expansions)
  – Regular Potential (Convergent Series);
  – Regular Potential (Asymptotic Series);
  – Singular Potential;

• Asymptotic Series

• Approaches for Selection of the Basis Functions
Far Field Expansions

(S-expansions)

Let

\[ \mathbf{x}_* \in \mathbb{R}^d. \]

We call expansion

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*) \]

far field expansion (or S-expansion) outside a sphere

\[ |y - x_*| > R_*, \]

if the series converges for \( \forall y, |y - x_*| > R_* \).
Far Field Expansion of a Regular Potential

...sometimes like this:

Can be like this:

...sometimes like this:

\[ |y - x*| > R* > |x_i - x*| \]

\[ |x_i - x*| > |y - x*| > R* \]
Local Expansion of a Regular Potential
Can be Far Field Expansion Also
(Repeat Example from Lecture 3)

Valid for any $r_∗ < \infty$, and $x_i$,

$$
\Phi(y,x_i) = e^{-(y-x_i)^2} = \sum_{m=0}^{\infty} a_m(x_i,x_*) S_m(y-x_*).
$$

We have

$$
e^{-(y-x_i)^2} = e^{-[y-x_*-(x_i-x_*)]^2} = e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} e^{2(x_i-x_*)(y-x_*)}
$$

$$
= e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} \sum_{m=0}^{\infty} \frac{2^m(x_i-x_*)^m(y-x_*)^m}{m!}
$$

Choose

$$
a_m(x_i,x_*) = e^{-(x_i-x_*)^2}(x_i-x_*)^m, \quad m = 0, 1, ...,
$$

$$
S_m(y-x_*) = e^{-(y-x_*)^2} \frac{2^m}{m!}(y-x_*)^m, \quad m = 0, 1, ...
$$
Asymptotic Series

\[ f(x, \varepsilon) = f_0(x) \varphi_0(\varepsilon) + f_1(x) \varphi_1(\varepsilon) + f_2(x) \varphi_2(\varepsilon) + \ldots = \sum_{n=0}^{\infty} f_n(x) \varphi_n(\varepsilon) \]

\[ \lim_{\varepsilon \to 0} \frac{\varphi_n(\varepsilon)}{\varphi_{n+1}(\varepsilon)} = 0. \]

Gauge functions

\[ f(x, \varepsilon) - \sum_{n=0}^{p-1} f_n(x) \varphi_n(\varepsilon) = O(f_p(x) \varphi_p(\varepsilon)) \]

The asymptotic expansion is *uniform* in domain \( x \in \Omega \) if

\[ \forall x \in \Omega, \quad \left| f(x, \varepsilon) - \sum_{n=0}^{p-1} f_n(x) \varphi_n(\varepsilon) \right| = O(\varphi_p(\varepsilon)). \]

Otherwise the asymptotic expansion is not uniform.
Examples of Uniform and Non-Uniform Expansions

Example of uniform expansion:

\[ f(x, \varepsilon) = \frac{1}{x + \varepsilon}, \quad x > 10 \]

\[ f(x, \varepsilon) = \frac{1}{x} \left(1 + \frac{\varepsilon}{x}\right)^{-1} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^n}{x^n} \]

Example of non-uniform expansion:

\[ f(x, \varepsilon) = e^{\varepsilon x}, \quad x \in \mathbb{R}^1 \]

\[ e^{\varepsilon x} = \sum_{n=0}^{\infty} \frac{\varepsilon^n x^n}{n!} \cdot \]

Prove that! (Hint: consider \( x \gg \varepsilon^{-1} \).)
Example of Far Field Expansion of a Regular Function (Using Asymptotic Series)

\[ \Phi(y, x_i) = \frac{1}{1 + (y - x_i)^2} = \frac{1}{1 + [y - x_* - (x_i - x_*)]^2} = \frac{1}{(y - x_*)^2} \frac{(y - x_*)^2}{1 + [y - x_* - (x_i - x_*)]^2}. \]

Let

\[ \epsilon = \frac{1}{y - x_*} \]

\[ \Phi(\epsilon, x_i - x_*) = \epsilon^2 \frac{1}{1 + \left[ \frac{1}{\epsilon} - (x_i - x_*) \right]^2} = \epsilon^2 \frac{1}{\epsilon^2 + (1 - \epsilon x)^2} = \epsilon^2 f(x, \epsilon), \quad x = x_i - x_* \]

\[ f(x, \epsilon) = \frac{1}{\epsilon^2 + (1 - \epsilon x)^2} = \sum_{n=0}^{\infty} f_n(x) \epsilon^n \]

\[ f_n(x) = \frac{1}{n!} \frac{\partial^n f(x, \epsilon)}{\partial \epsilon^n} \Bigg|_{\epsilon=0} \]
Example of Far Field Expansion of a Regular Function (continuation)

\[ f_0(x) = 1, \]
\[ f_1(x) = \frac{\partial f(x, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} = 2x, \]
\[ f_2(x) = \frac{1}{2!} \frac{\partial^2 f(x, \epsilon)}{\partial \epsilon^2} \bigg|_{\epsilon=0} = 3x^2 - 1, \]

\[ \Phi(y, x_i) = \frac{1}{(y - x_*)^2} \sum_{n=0}^{\infty} f_n(x_i - x_*) \frac{1}{(y - x_*)^n} \]

\[ y \geq 100, \quad x_i = 1, \quad x_* = 0, \]
\[ \epsilon = 10^{-2}, \quad x = 1 \]

\[ \left| \Phi(y, x_i) - \frac{1}{(y - x_*)^2} \left[ 1 + \frac{2(x_i - x_*)}{(y - x_*)} \right] \right| \leq \epsilon^4 (3x^2 - 1) = 2 \cdot 10^{-8}. \]
Far Field Expansion of a Singular Potential

...sometimes like this:

\[ |y - x_*| > R_* > |x_i - x_*| \]

...sometimes like this:

\[ |x_i - x_*| > |y - x_*| > R_* \]

Can be like this:

\[ |y - x_*| > R_* \geq |x_i - x_*| \]

This case only!
Example For S-expansion of Singular Potential

\[ \Phi(y, x_i) = \frac{1}{y - x_i}. \]

\[ \frac{1}{y - x_i} = \frac{1 - \frac{x_i - x_i}{y - x_i}}{(y - x_*) \left[ 1 - \frac{x_i - x_+}{y - x_*} \right]} = \frac{1}{(y - x_*) \left[ 1 - \frac{x_i - x_*}{y - x_*} \right]}^{-1}. \]

\[ \left[ 1 - \frac{x_i - x_*}{y - x_*} \right]^{-1} = \sum_{m=0}^{\infty} \frac{(x_i - x_*)^m}{(y - x_*)^m}, \quad |y - x_*| > |x_i - x_*|. \]

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*), \]

\[ b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \ldots, \]

\[ S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \ldots \]
Let us compare with the R-expansion of the same function

\[ |y - x_*| < |x_i - x_*| : \]

**R-expansion**

\[
\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y-x_*),
\]

\[
a_m(x_i, x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, \ldots,
\]

\[
R_m(y-x_*) = (y - x_*)^m, \quad m = 0, 1, \ldots
\]

\[ |y - x_*| > |x_i - x_*| : \]

**S-expansion**

\[
\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y-x_*),
\]

\[
b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \ldots,
\]

\[
S_m(y-x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \ldots
\]

**Singular Point is located at the Boundary of regions for the R- and S-expansions!**
What Do We Need For Real FMM (that provides spatial grouping)

We need S-expansion for $|y - x_*| > R_* > |x_i - x_*|$
We need R-expansion for $|y - x_*| < r_* < |x_i - x_*|$
Basis Functions

• Power series are great, but do they provide the best approximation? (sometimes yes!)

• Other approaches to factorization:
  – Asymptotic Series (Can be divergent!);
  – Orthogonal Bases in $L_2$;
  – Eigen Functions of Differential Operators;
  – Functions Generated by Differentiation or Other Linear Operators.

• Some of this approaches will be considered in this course.