Compute the matrix-vector product

$$\mathbf{v} = \mathbf{\Phi u}, \tag{1}$$

or

$$v_j = \sum_{i=1}^{N} \Phi_{ji} u_i, \quad j = 1, ..., M, \tag{2}$$

with absolute error $\epsilon < 10^{-6}$.

where

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_{11} & \Phi_{12} & ... & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & ... & \Phi_{2N} \\ ... & ... & ... & ... \\ \Phi_{M1} & \Phi_{M2} & ... & \Phi_{MN} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_N \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ ... \\ v_M \end{pmatrix}, \tag{3}$$

$$\Phi_{ji} = \frac{1}{y_j - x_i}, \quad i = 1, ..., N, \quad j = 1, ..., M.$$

and $x_1, ..., x_N$ are random points uniformly distributed on [0,10], $M = N - 1$, and each y_j is located between the closest x_i's on each side, $j = 1, ..., N - 1$.

1. Check if your program for S|S and R|R operators works correctly, by comparing it with a direct evaluation for a single source; and then evaluated via a S expansion, followed by a S|S translation; and evaluated via an R expansion followed by a R|R translation. (You already should have checked the S expansion, R expansion, and S|R translation routines in previous homework).
2. Write a program that implements both straightforward multiplication based on Eq. (2) and Multi Level FMM (MLFMM).
3. Provide a graph of the absolute maximum error between the straightforward and the MLFMM method for $N = 10^3$, and grouping parameter s varying between 1 and 100, and several p ($p \sim 10$).
4. Provide a graph of the CPU time vs s at fixed p that insures that the required accuracy is achieved. Find optimum s for your implementation.
5. Provide a graph that compares the CPU time required by the straightforward method and the MLFMM for N varying between 10^2 and 10^3 for straightforward and N varying between 10^2 and 10^4 for the MLFMM (use the optimum s found).
6. Find the "break-even" point (i.e. N at which the "Fast" method requires the same CPU time as the straightforward method) for your implementation.
7. Provide a graph of actual error (between the standard and the fast methods) for N varying between 10^2 and 10^3 and the truncation numbers used.
8. Compare the performance of the MLFMM with the SLFMM (developed in the previous HW).

Hints

Use your previous homework programs to set data structure, compute S|R-translation operators, and straightforward solution.

Separate the part that sets data structure from the run part of the MLFMM. For comparisons measure only the CPU time required for run part of the MLFMM. You can also compare the efficiency of your program that sets data structure by comparison of the CPU time of the two parts of the MLFMM.