

### Homework 4: CMSC 878R/AMSC698R

Evaluate the accuracy of the  $p$ -truncated  $S|R$ -reexpansion of function with translation vector  $t$  (in the 1-D case this is a scalar)

$$\Phi(y, x_i) = \frac{1}{y - x_i},$$

where  $x_i$  is an arbitrary point,  $x_i \in [-\frac{1}{2}l, \frac{1}{2}l]$ ,  $y \in [\frac{3}{2}l, \frac{5}{2}l]$ ,  $t = 2l$ .

1. Obtain a formula for the singular (or multipole)  $S$ -expansion of  $\Phi(y, x_i)$  near  $x_* = 0$ , that can be used for approximation of this function for  $|y| \geq \frac{3}{2}l$ . Find the dependence of the maximum absolute error  $\epsilon$  on the truncation number  $p$  and the length of the interval  $l$ . Also provide a formula for the inverse function, i.e. the  $p(l, \epsilon)$ , which enables determining the truncation number for given  $l$  and  $\epsilon$ . Compare your theoretical dependence with numerical tests (e.g. you can take  $x_i = \frac{1}{2}l$ ;  $y = \frac{3}{2}l$ , and vary  $p$  and  $l$ ) and provide a plot of the comparison.
2. Implement a routine that provides  $p$ -truncated  $S|R$ -translation of the  $S$ -expansion coefficients to  $R$ -expansion coefficients (so it can be used in a future FMM program). As input your routine take a vector of  $S$ -expansion coefficients of length  $p$  and translation vector  $t$ , and as output it should provide a vector of  $R$ -expansion coefficients at the new center, of length  $p$ .
3. Test your program using as input coefficients of the  $p$ -truncated  $S$ -expansion of  $\Phi(y, x_i)$  and vector  $t = 2l$ . Convolve the output vector of  $R$ -expansion coefficients with  $R$ -basis functions centered at  $x_* = 2l$  to get approximate value of  $\Phi(y, x_i)$  at  $y \in [\frac{3}{2}l, \frac{5}{2}l]$ . Compare this result with exact (straightforward) computation of  $\Phi(y, x_i)$ , and with the  $S$  expansion.
4. Obtain a theoretical error bounds of  $p$ -truncated  $S|R$ -translation. Compare results with your numerical experiments at various  $p$  and  $l$ . Provide a graph that illustrates and compares theoretical and numerical dependences. Make some conclusions.

#### Hints

1. Use the property of geometric progression for error evaluations

$$\frac{1}{1 - \alpha} = \frac{1 - \alpha^p + \alpha^p}{1 - \alpha} = \frac{1 - \alpha^p}{1 - \alpha} + \frac{\alpha^p}{1 - \alpha} = 1 + \alpha + \dots + \alpha^{p-1} + \frac{\alpha^p}{1 - \alpha}, \quad (\alpha \neq 1). \quad (1)$$

(if you wish, you still can use the Taylor series and evaluation of its residual term).

2. Use Matlab.
4. Consider  $|\Phi(y, x_i) - \Phi^{(p)}(y, x_i)|$ , where  $\Phi^{(p)}(y, x_i)$  is obtained by application of the  $S|R$ -truncated matrix to the  $p$ -truncated vector of the  $S$ -expansion coefficients and convolved with the  $R$ -basis functions. The series will contain finite number of terms. At the same time  $\Phi(y, x_i)$  will be represented by the same series with an infinite number of terms, so you need to evaluate the ‘tails’ (which will be neglected). You may use the fact that the series converges absolutely and uniformly, and use the inequality

$$\left| \sum_n a_n \right| \leq \sum_n |a_n|, \quad (2)$$

and the following the generalization of Equation (1):

$$\frac{1}{(1 - \alpha)^{m+1}} = \sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!} \alpha^n, \quad |\alpha| < 1, \quad m = 0, 1, \dots \quad (3)$$