In preparation for starting the FMM, we will consider the problem of translating series, and evaluating the error bounds of this translation. In addition, please read the handout given in class.

1 Problem (Homework 4)

Evaluate accuracy of the p-truncated $S|R$-reexpansion of function with translation vector t (in 1-D case this is a scalar)

$$\Phi(y, x_i) = \frac{1}{y - x_i},$$

where x_i is an arbitrary point, $x_i \in \left[-\frac{1}{2}l, \frac{1}{2}l\right]$, $y \in \left[\frac{3}{2}l, \frac{5}{2}l\right]$, $\quad t = 2l$.

Homework 4

1. Using examples of Lectures #5 and 8, provide a formula for S-expansion of $\Phi(y, x_i)$ near $x_* = 0$, that can be used for approximation of this function for $|y| \geqslant \frac{3}{2}l$. Find theoretically dependence of the maximum absolute error ϵ on truncation number p and the length of the interval l. Compare your theoretical dependence with some computations (e.g. you can take $x_i = \frac{1}{2}l$; $y = \frac{3}{2}l$, and vary p and l) and provide a graph of comparisons.

2. Implement a routine that provides p-truncated $S|R$-translation matrix that converts a $p$vector of S-expansion coefficients to a p vector of R-expansion coefficients, as was discussed in class in lecture 8. This will eventually be used to develop a future FMM program). As input your routine should take a vector of S-expansion coefficients of length p and translation vector t (in 1-D t is a scalar) and as output it should provide a vector of R-expansion coefficients of length p.

3. Test your program using as input coefficients of the p-truncated S-expansion of $\Phi(y, x_i)$ and vector $t = 2l$. Using the the output vector of R-expansion coefficients and the R-basis functions centered at $x_* = 2l$, obtain the approximate value of $\Phi(y, x_i)$ for $y \in \left[\frac{3}{2}l, \frac{5}{2}l\right]$. Compare this result with the exact (straightforward) computation of $\Phi(y, x_i)$.

4. Obtain theoretical error bounds for p-truncated $S|R$-translations. Compare the results of your numerical experiments at various p and l with this bound, and plot. Make some conclusions.

Hints (Homework 4)

1. Use the property of geometric progression for error evaluation

$$\frac{1}{1-\alpha} = \frac{1-\alpha^p+\alpha^p}{1-\alpha} = \frac{1-\alpha^p}{1-\alpha} + \frac{\alpha^p}{1-\alpha} = 1 + \alpha + ... + \alpha^{p-1} + \frac{\alpha^p}{1-\alpha}, \quad (\alpha \neq 1). \tag{1}$$

(if you wish, you can also use the technique taught previously – Taylor series and evaluation of the residual term).

2. Use Matlab.

3. In 4 above, note that the norm of the exact translation operator for bounded functions does not exceed 1. You may use this fact and the Theorem proved in class in lecture #7 to evaluate the error bounds of the p-truncated translation operator. On the other hand you have the freedom to evaluate these bounds in a way you devise (any method is OK).