- Error estimate for one R-expansion. If we have K boxes, then the size of the box is $l = 1/K$. The distance from the center of the box to the closest source outside the neighborhood is then $3l/2 = 3/(2K)$. We have the following factorization:

$$\begin{aligned}
\Phi(y, x_i) &= \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y - x_*), \\
a_m(x_i, x_*) &= -(x_i - x_*)^{-m-1}, \quad m = 0, 1, ..., \\
R_m(y - x_*) &= (y - x_*)^m, \quad m = 0, 1, ...
\end{aligned}$$

Here

$$|y - x_*| \leqslant \frac{l}{2}, \qquad |x_i - x_*| \geqslant \frac{3l}{2}$$

Truncation error of the exact expansion is then

$$\begin{aligned}
\epsilon_p &= \left| \Phi(y, x_i) - \sum_{m=0}^{p-1} a_m(x_i, x_*) R_m(y - x_*) \right| = \left| \sum_{m=p}^{\infty} a_m(x_i, x_*) R_m(y - x_*) \right| = \left| -\frac{1}{x_i - x_*} \sum_{m=p}^{\infty} \left(\frac{y - x_*}{x_i - x_*} \right)^m \right| \\
&= \left| \frac{1}{y - x_i} \left(\frac{y - x_*}{x_i - x_*} \right)^p \right| = \left| \frac{1}{y - x_i} \right| \left| \left(\frac{y - x_*}{x_i - x_*} \right)^p \right| \leqslant \frac{1}{l} \frac{1}{3^p},
\end{aligned}$$

since

$$|y - x_i| \leqslant l.$$

- The total error of the PreFMM is

$$\epsilon < N\epsilon_p \leqslant \frac{N}{l} \frac{1}{3^p} = \frac{NK}{3^p}.$$

So for given ϵ and p :

$$p \geqslant \frac{\log(NK) - \log(\epsilon)}{\log 3}$$

we achieve the required accuracy. For $N = 10^3$, $K = 10^2$ and $\epsilon = 10^{-6}$ we have

$$p \geqslant \frac{\log 10^5 - \log 10^{-6}}{\log 3} \approx 23.05.$$

- Theoretical optimum K is

$$K_{opt} \sim \sqrt{\frac{MPow(1)}{p}} < \sqrt{\frac{2N}{p}}$$

While this number depends on p, for $p \geqslant 2$, we should have

$$K_{opt} \lesssim N^{1/2}.$$

The maximum N is 10^4 and so the maximum K should not exceed 10^2. In any case for $\epsilon = 10^{-6}$:

$$\max \frac{\log(NK) - \log(\epsilon)}{\log 3} = \frac{\log 10^6 - \log 10^{-6}}{\log 3} \approx 25.15.$$

- Computations with $p = 24$ for a range $10 < K < 100$ and $N = 1000$ show that the actual error is much smaller than the theoretical error and smaller p can be used.

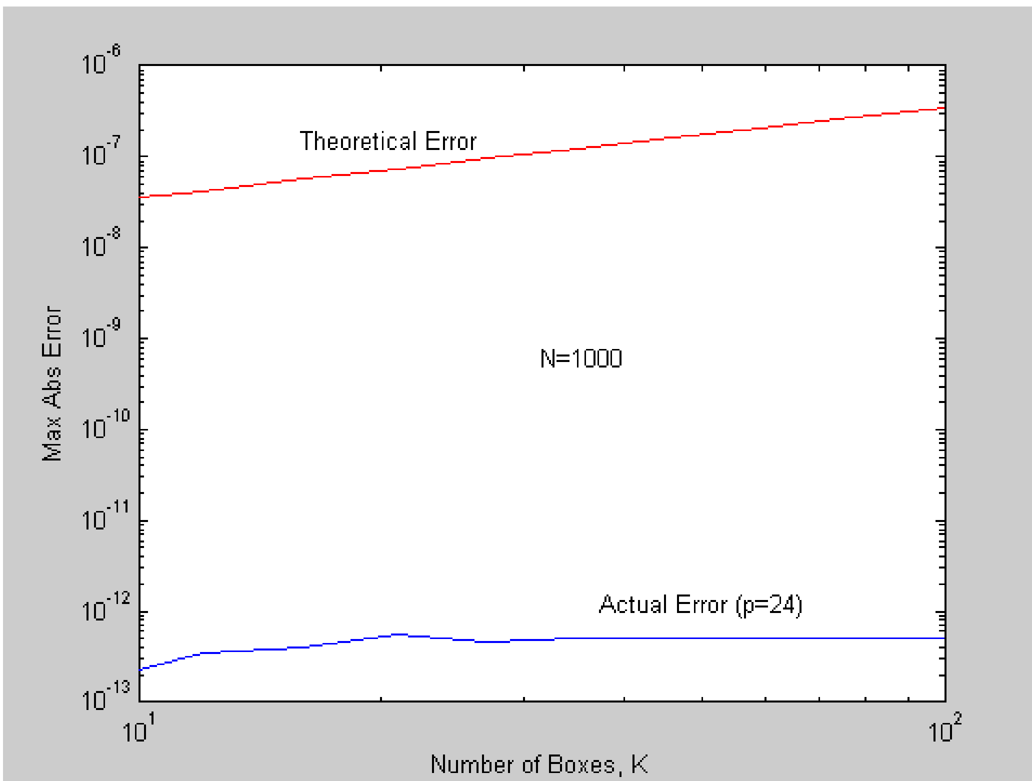

Theoretical Error
N=1000
Actual Error (p=24)
Max Abs Error
Number of Boxes, K
10^-6
10^-7
10^-8
10^-9
10^-10
10^-11
10^-12
10^-13
10^1
10^2

Figure 1:

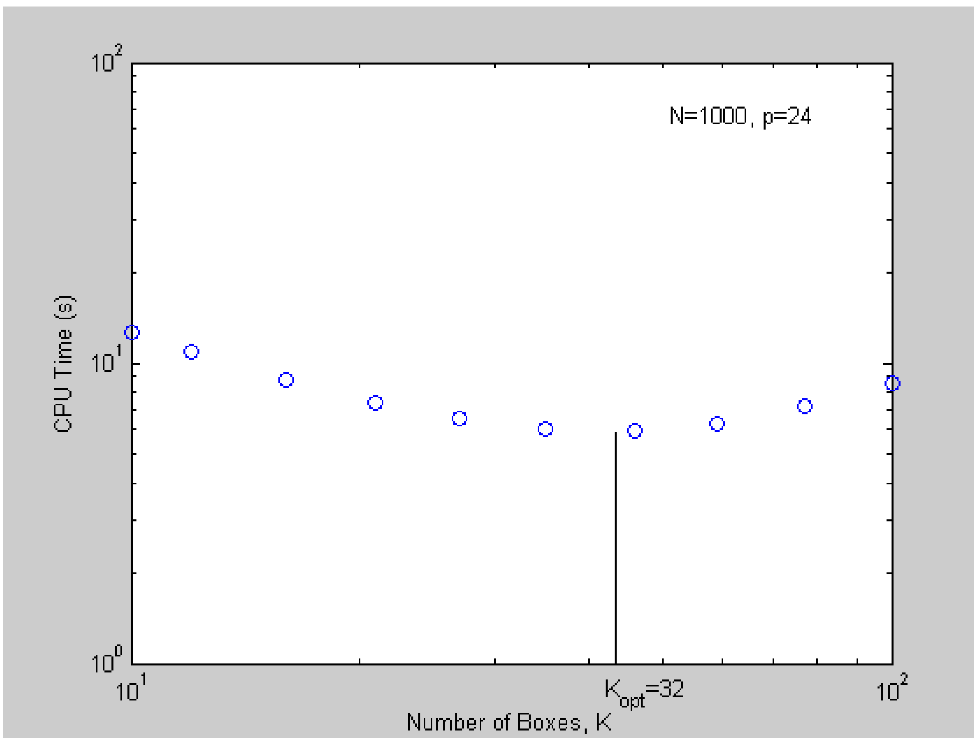


Figure 2:

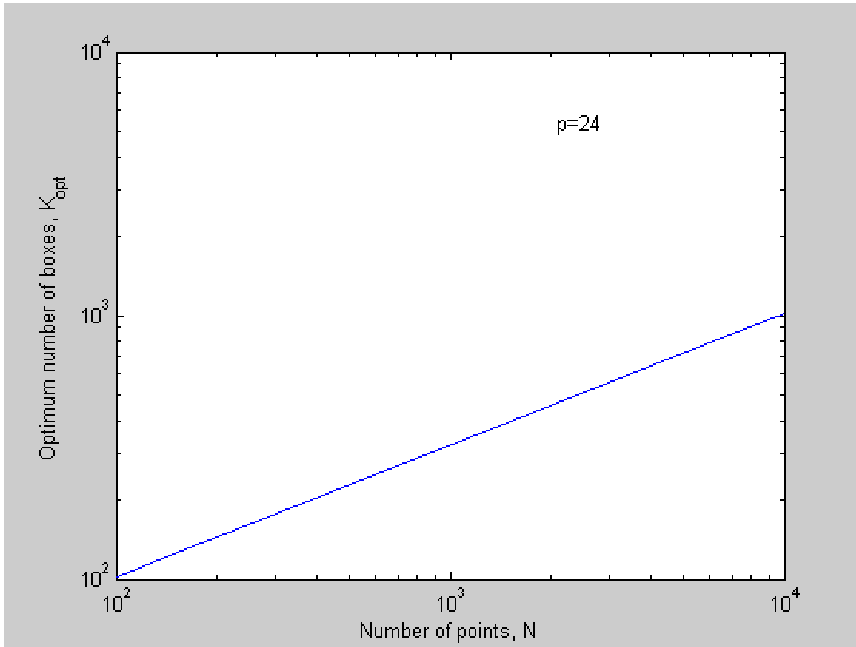


Figure 3:

- Computations with $p = 24$ for a range $10 < K < 100$ and $N = 1000$ show that there exists a minimum of CPU time for $K_{opt} \approx 32$.Assuming that

$$K_{opt} = AN^{1/2}$$

for optimum K at sufficiently large N we can find that for our computations

$$A \sim \frac{32}{1000^{1/2}} \approx 10.$$

This dependence is plotted belowWe also can scale this as

$$K_{opt} = B\left(\frac{N}{p}\right)^{1/2}, \; B \sim Ap^{1/2} \approx 5 \cdot 10 = 50.$$

- The error for computation with $p = 24$ and $K = K_{opt}(N)$ is plotted below:
- Computations with the required accuracy can be also performed with lower p, e.g. $p = 15$:But this should not increase the speed substantially, since complexity is $O(p^{1/2})$.
- Required dependences are plotted:

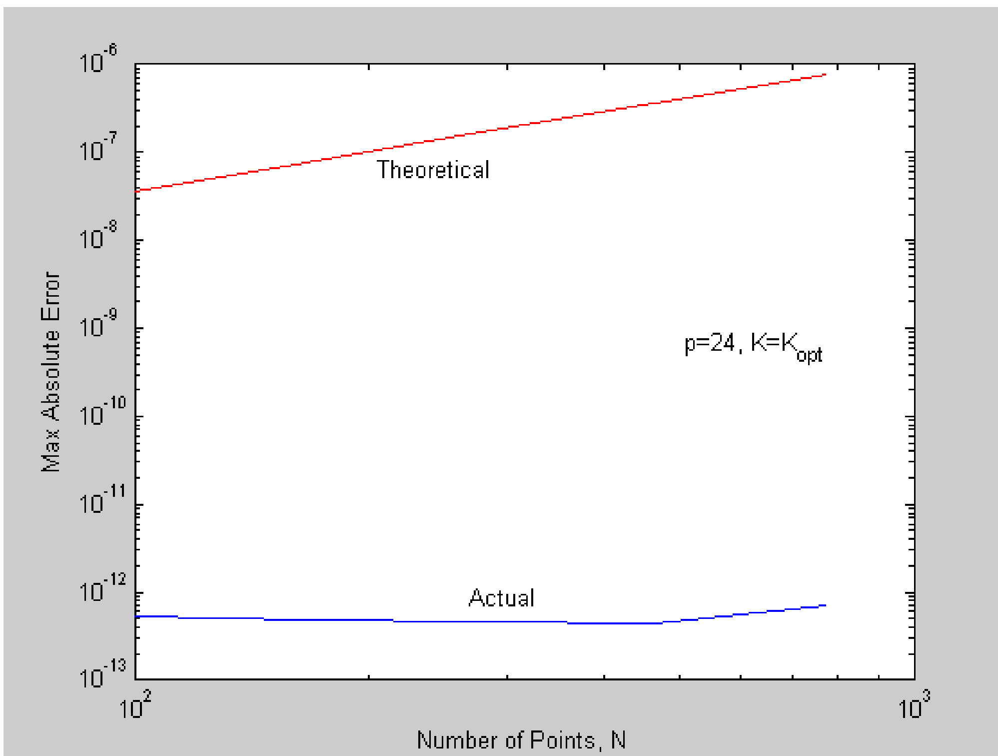

Theoretical
Actual
p=24, K=K_opt
Max Absolute Error
Number of Points, N
10^-6
10^-7
10^-8
10^-9
10^-10
10^-11
10^-12
10^-13
10^2
10^3

Figure 4:

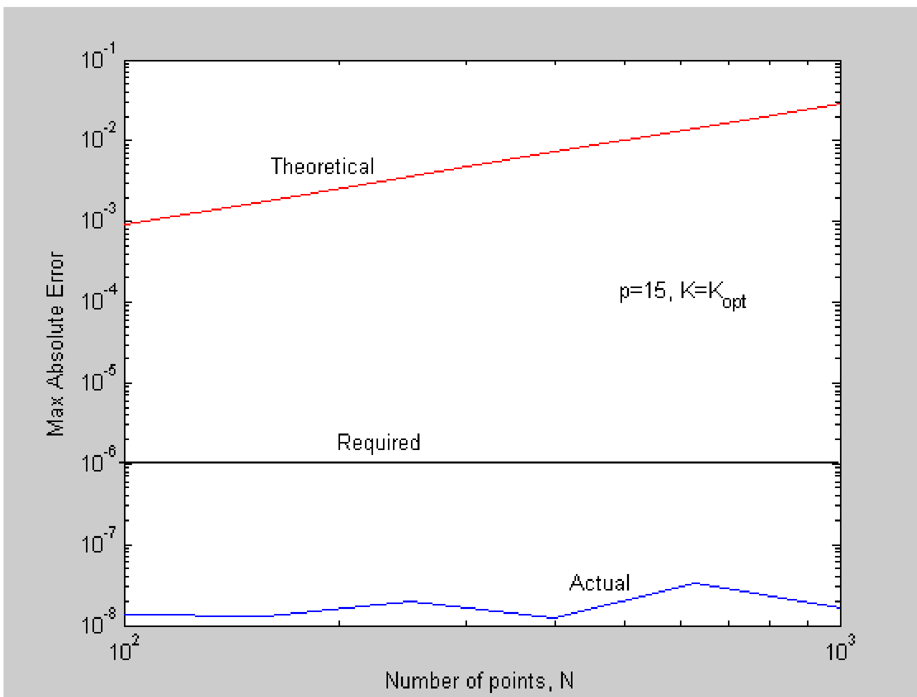
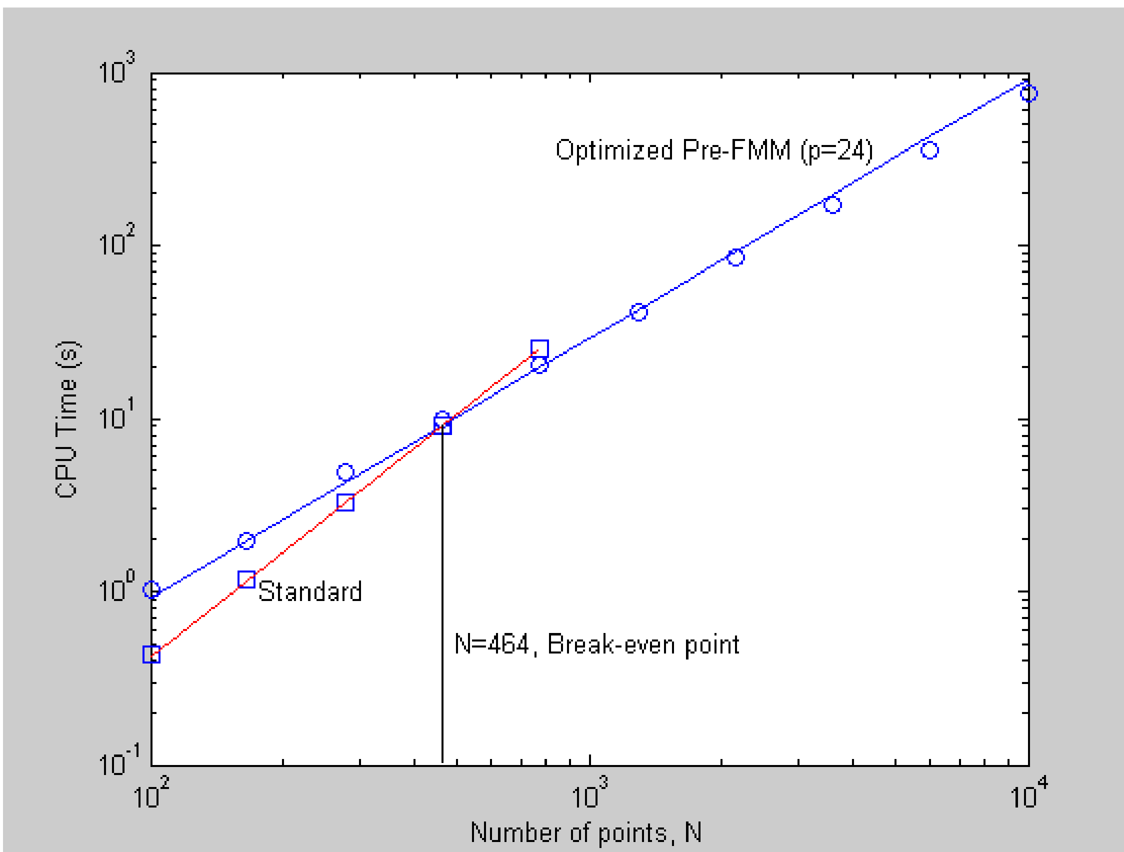

Theoretical
Required
Actual
p=15, K=K_opt
Max Absolute Error
Number of points, N
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
10^2
10^3

Figure 5:



Optimized Pre-FMM (p=24)
Standard
N=464, Break-even point
CPU Time (s)
Number of points, N
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{2}
10^{3}
10^{4}

Figure 6: