

1 Problem (Homework 2)

This week's assignment uses the ideas of using Taylor series to achieve factorizations suitable for use in FMM type algorithms.

Let

$$\Phi_{jk} = e^{i\alpha x_k y_j}, \quad k = 1, \dots, N, \quad j = 1, \dots, M.$$

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2N} \\ \dots & \dots & \dots & \dots \\ \Phi_{M1} & \Phi_{M2} & \dots & \Phi_{MN} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_N \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \dots \\ v_M \end{pmatrix}, \quad (1)$$

where $i^2 = -1$, $x_1, \dots, x_N, y_1, \dots, y_M, u_1, \dots, u_N$, are random real numbers distributed uniformly in $[0, 1]$, and α is some real number. Compute the matrix-vector product

$$\mathbf{v} = \Phi \mathbf{u}, \quad \text{or} \quad v_j = \sum_{k=1}^N \Phi_{jk} u_k, \quad j = 1, \dots, M, \quad (2)$$

with absolute error $\epsilon < 10^{-6}$. The matrix sizes, $N, M > 0$ are given (fixed) positive integers.

1. Write down a factored expression. Estimate the error in truncating the series using residual term evaluation for the Taylor series, and evaluate the truncation number, p , as a function of the required accuracy and N . Provide a formula that can be used for the "fast" ($O(N + M)$) method.
2. Write a program that implements both the straightforward computation based on Eq. (2) and the "fast" method.
3. Plot the absolute maximum error between the straightforward and "Fast" method for $N = 10^3$ and $M = 2N$ and p varying between 1 and 11 for $\alpha = 1$ and $\alpha = 2$. Compare the results with your evaluations of the accuracy.
4. Provide a graph that compares the CPU time required by the straightforward and the "Fast" method for N varying between 10^2 and 10^3 for the straightforward and N varying between 10^2 and 10^4 for the "Fast" method. Take $M = 2N$ and the theoretical value of the truncation number that ensures that the required accuracy is achieved for $\alpha = 1$.
5. Provide a graph of the abs. max. error (between the standard and fast methods) for N varying between 10^2 and 10^3 , $M = 2N$ and the truncation numbers used for each N for $\alpha = 1$.

Hints.

1. Note that each source contributes to the error. So the truncation number p , corresponding to a required accuracy ϵ , depends on N . This relationship is an implicit function of p and you can either solve for p (write a Matlab function to do that) or determine it by developing a table of values and interpolating.
2. The maximum absolute error is defined as

$$error = \max_{i=1, \dots, N} |v_i^{straightforward} - v_i^{fast}|. \quad (3)$$

Plot the theoretical error bound on the same graph (use hint 1).

3. You may keep the truncation number constant (using the one evaluated for $N \leq 10^4$) or vary it with N according to the theoretical estimate for the error. In this case the function calculated in hint 1 will be helpful.