CMSC698R/AMSC878R Mid Term Examination

1. (30 points) Below are some very short questions (mainly definitions to check if you have read the notes). Be concise in answering these questions, because you will need time to do Questions 2 and 3.

   (a) (5 points) Show that the product of a square matrix, whose entries have a degenerate factorization
   $$\Phi(x_i, y_j) = \sum_{k=0}^{p-1} \psi_k(x_i)\,\phi_k(y_j), \qquad i, j = 1, \ldots, N, \quad p = o(N)$$
   with a vector can be done in $o(N^2)$ operations.

   (b) (5 points) What is a local expansion? What is a far-field expansion? What kind of an expansion is a Taylor series?

   (c) (5 points) What are multinomial coefficients? How can you use them to reduce the number of terms in multidimensional power series?

   (d) (7 points) Let $\Phi(y, x)$ be a function. What are the requirements on Φ so that the sums
   $$v(y_j) = \sum_{i=1}^{N} \Phi(y_j, x_i)\, u_i, \qquad j = 1, \ldots, M$$
   can be evaluated using the FMM?

   (e) (8 points) What is bit-interleaving? How is it applied to find the index of a box containing a given point?

2. (45 points) Consider the pre-FMM algorithm with R-expansions that was discussed in class. We use it to compute the sum
   $$v(y_j) = \frac{1}{N} \sum_{i=1}^{N} \Phi(y_j, x_i)\, u_i, \qquad j = 1, \ldots, N$$
   The points $\{y_j\}$ and $\{x_i\}$ are distributed uniformly in a 2-D square domain of unit size. The domain is divided into K equal square boxes. It turns out that for this $\Phi(y_j, x_i)$ the number of terms, p, required for a given error, ϵ, in the R-expansion is proportional to the distance of the evaluation points from the expansion center.

   (a) Derive an expression for p as a function of K for fixed ϵ, by using the maximum distance in the evaluation box.

   (b) Derive an expression for the complexity of the algorithm as a function of N and K. Assume $K \gg 1$.

   (c) Determine the optimal number of boxes K as a function of N, and the overall complexity of the algorithm.

3. (25 points) Find the center of box #1624 (decimal) at level 6 of the quadtree.

