

FMM CMSC 878R/AMSC 698R

Lecture 7

Outline

- Representation of functions in the space of coefficients
- Matrix representation of operators
- Truncation and truncated operators
- Translation operator
- Reexpansion coefficients
- $R|R$ and $S|S$ translation operators
- Examples
- $S|R$ and $R|S$ translation operators
- Properties of translation operators

Why do we need represent functions in different spaces?

- Functions should be efficiently summed up;
- Sums of functions should be compressed;
- Error bounds should be established;
- Functions should be translated and expanded over different bases;
- For computations we need discrete and finite function representations.
- Some functions measured experimentally or approximated by splines, and there is no explicit analytical representation in the whole space.

Function Representation in the Space of Coefficients

Let $\mathbb{F}(\Omega) \subset C(\Omega)$, $\Omega \subset \mathbb{R}^d$, be a normed space of continuous functions with norm

$$\|\Phi(\mathbf{y})\| = \max_{\mathbf{y} \in \Omega} |\Phi(\mathbf{y})|.$$

Let also $\{F_n(\mathbf{y})\}$ be a complete basis in $\mathbb{F}(\Omega)$, so

$$\Phi(\mathbf{y}) = \sum_{n=0}^{\infty} A_n F_n(\mathbf{y}), \quad \mathbf{y} \in \Omega \subset \mathbb{R}^d, \quad \Phi(\mathbf{y}), F_n(\mathbf{y}) \in \mathbb{F}(\Omega),$$

absolutely and uniformly converges in $\Omega \subset \mathbb{R}^d$. This means that

$$\forall \epsilon > 0, \quad \exists p(\epsilon), \quad |\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})| < \epsilon, \quad \forall \mathbf{y} \in \Omega,$$

$$\forall \epsilon > 0, \quad \exists p(\epsilon), \quad \sum_{n=p}^{\infty} |A_n F_n(\mathbf{y})| < \epsilon, \quad \forall \mathbf{y} \in \Omega,$$

$$\Phi^p(\mathbf{y}) = \sum_{n=0}^{p-1} A_n F_n(\mathbf{y}).$$

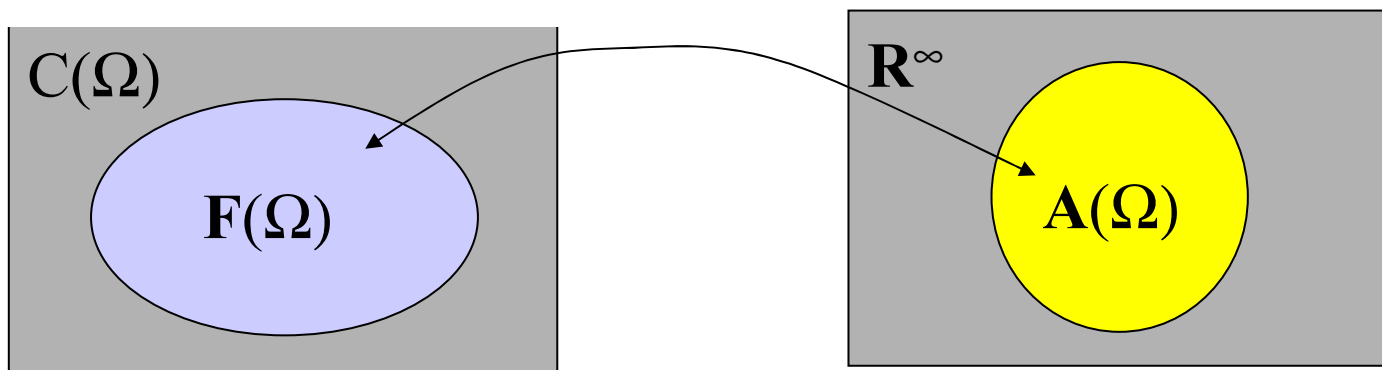
Function Representation in the Space of Coefficients (2)

Expansion coefficients can be stacked in an infinite vector

$$\mathbf{A} = \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_n \\ \dots \end{pmatrix}.$$

Let us denote $\mathbb{A}(\Omega)$ a subset of \mathbb{R}^∞ which is an image of $\mathbb{F}(\Omega)$. For any $\mathbf{A} \in \mathbb{A}(\Omega)$ we request that there exists one-to-one mapping

$$\Phi(\mathbf{y}) \ni \mathbf{A}, \quad \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \quad \mathbf{A} \in \mathbb{A}(\Omega) \subset \mathbb{R}^\infty.$$



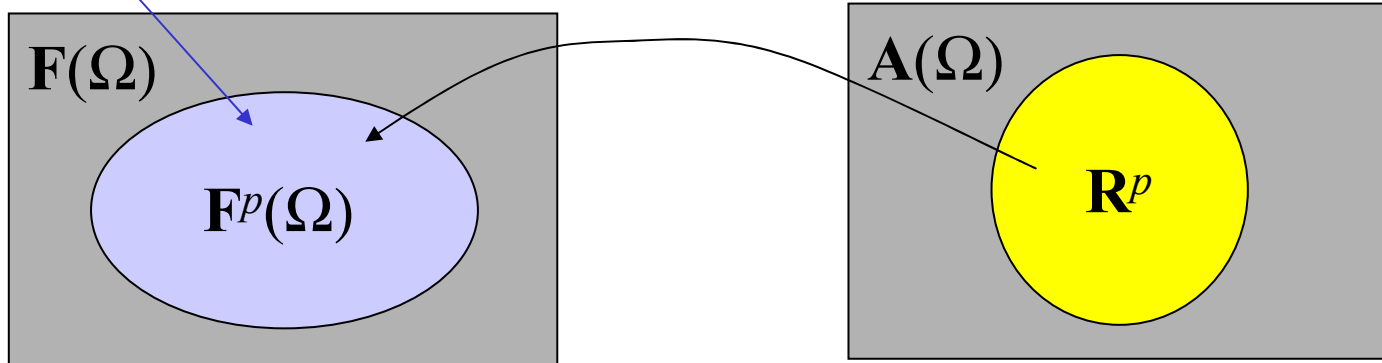
p-Truncated Vectors

$$\forall \mathbf{A} \in \mathbb{R}^p, \quad \exists \Phi^p(\mathbf{y}) = \sum_{n=0}^{p-1} A_n F_n(\mathbf{y}) \in \mathbb{F}^p(\Omega) \subset \mathbb{F}(\Omega).$$

$\mathbb{F}^p(\Omega)$ is dense in $\mathbb{F}(\Omega)$:

$$\forall \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \quad \exists p, \Phi^p(\mathbf{y}) \in \mathbb{F}^p(\Omega), \quad \|\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})\| = \max_{\mathbf{r} \in \Omega} |\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})| < \epsilon.$$

Dense in $\mathbb{F}(\Omega)$



Matrix Representation of Linear Operators

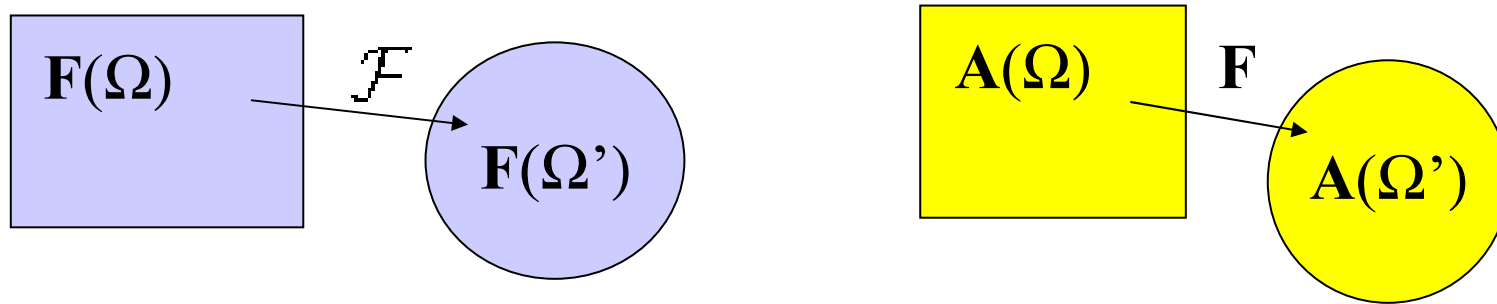
Let $\Omega' \subset \Omega$ and \mathcal{F} is a mapping of $\mathbb{F}(\Omega)$ to $\mathbb{F}(\Omega')$. Such mapping can be considered as action of operator \mathcal{F} on Φ :

$$\mathcal{F}[\Phi(\mathbf{y})] = \widetilde{\Phi(\mathbf{y})}, \quad \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \quad \widetilde{\Phi(\mathbf{y})} \in \mathbb{F}(\Omega') \subset \mathbb{F}(\Omega)$$

Respectively, operator \mathcal{F} generates operator \mathbf{F} that maps the space of expansion coefficients $\mathbb{A}(\Omega) \rightarrow \mathbb{A}(\Omega')$, which can be considered as *representation* of the operator \mathcal{F} in the space of expansion coefficients:

$$\mathbf{F}\mathbf{A} = \widetilde{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{A}(\Omega), \quad \widetilde{\mathbf{A}} \in \mathbb{A}(\Omega') \subset \mathbb{A}(\Omega).$$

Inversly, if we introduce any transform of expansion coefficients $\mathbf{F}\mathbf{A} = \widetilde{\mathbf{A}}$ which provides uniform convergence of function $\widetilde{\Phi(\mathbf{y})}$ corresponding to these coefficients in $\Omega' \subset \Omega$ then such transform can be treated as operator \mathcal{F} that convert one function from $\mathbb{F}(\Omega)$ to another.



Representation of a Linear Operator

p-Truncation (Projection) Operator

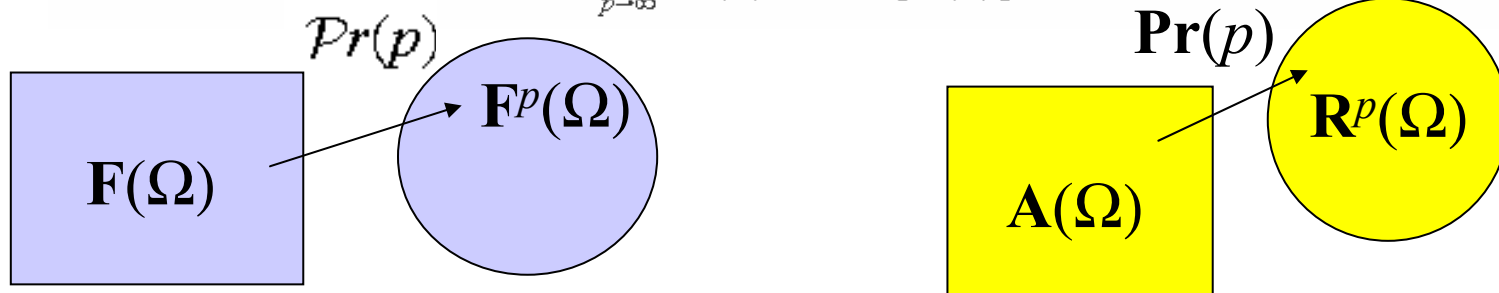
$$\Pr(p)\mathbf{A} = \tilde{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{A}(\Omega), \quad \tilde{\mathbf{A}} \in \mathbb{A}^p(\Omega).$$

$$\mathbf{A} = \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_{p-1} \\ A_p \\ A_{p+1} \\ \dots \end{pmatrix} \rightarrow \tilde{\mathbf{A}} = \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_{p-1} \\ 0 \\ 0 \\ \dots \end{pmatrix}, \quad \tilde{\mathbf{A}} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \mathbf{A}$$

In space $\mathbb{F}(\Omega)$:

$$\Pr(p)[\Phi(\mathbf{y})] = \Phi^p(\mathbf{y}), \quad \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \quad \Phi^p(\mathbf{y}) \in \mathbb{F}^p(\Omega),$$

$$\lim_{p \rightarrow \infty} \|\Phi(\mathbf{y}) - \Pr(p)[\Phi(\mathbf{y})]\| = 0.$$



Norm of p-Truncation Operator (important for error bounds)

Norm:

$$\|Pr(p)\| = \frac{\sup_{\mathbf{y} \in \Omega} \|Pr(p)[\Phi(\mathbf{y})]\|}{\sup_{\mathbf{y} \in \Omega} \|\Phi(\mathbf{y})\|}.$$

Triangle inequality:

$$\|I\| - \|I - Pr(p)\| \leq \|Pr(p)\| \leq \|I\| + \|I - Pr(p)\| = 1 + \|I - Pr(p)\|$$

$$\forall \epsilon > 0, \exists p, \|I - Pr(p)\| < \epsilon,$$

so

$$\forall \epsilon > 0, \exists p, 1 - \epsilon < \|Pr(p)\| < 1 + \epsilon,$$

p-Truncated Operator

Let $H : F(\Omega) \rightarrow F(\Omega)$ be an operator, that is represented by infinite matrix

$$H = \begin{pmatrix} h_{00} & h_{01} & \dots & h_{0,p-1} & h_{0p} & \dots \\ h_{10} & h_{11} & \dots & h_{1,p-1} & h_{1p} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_{p-1,0} & h_{p-1,1} & \dots & h_{p-1,p-1} & h_{p-1,p} & \dots \\ h_{p0} & h_{p1} & \dots & h_{p-1,p} & h_{pp} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

We call operator $H^{(p)} : F(\Omega) \rightarrow F(\Omega)$, *p-truncated* if it is represented by matrix

$$H^{(p)} = \begin{pmatrix} h_{00} & h_{01} & \dots & h_{0,p-1} & 0 & \dots \\ h_{10} & h_{11} & \dots & h_{1,p-1} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_{p-1,0} & h_{p-1,1} & \dots & h_{p-1,p-1} & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Norm of p -Truncated Operator (important for error bounds)

Theorem: Let $H : F(\Omega) \rightarrow F(\Omega)$, such that $0 < \|H\| < \infty$, and $H^{(p)} : F(\Omega) \rightarrow F(\Omega)$ is the p -truncated operator H . Let also $p(\epsilon)$ be such that $1 - \epsilon < \|Pr(p)\| < 1 + \epsilon$. Then

$$(1 - \epsilon)^2 < \|Pr(p)\|^2 = \frac{\|H^{(p)}\|}{\|H\|} = \|Pr(p)\|^2 < (1 + \epsilon)^2,$$

$$\lim_{p \rightarrow \infty} \frac{\|H^{(p)}\|}{\|H\|} = 1.$$

Proof.

A p -truncated operator can be represented in the form

$$H^{(p)} = Pr(p)HPr(p)$$

(check!)

So the norm of $H^{(p)}$ is

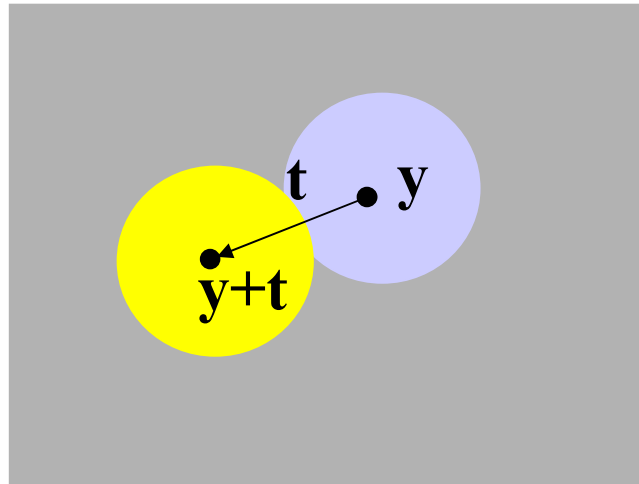
$$\|H^{(p)}\| = \|Pr(p)\| \|H\| \|Pr(p)\| = \|H\| \|Pr(p)\|^2.$$

End of Proof.

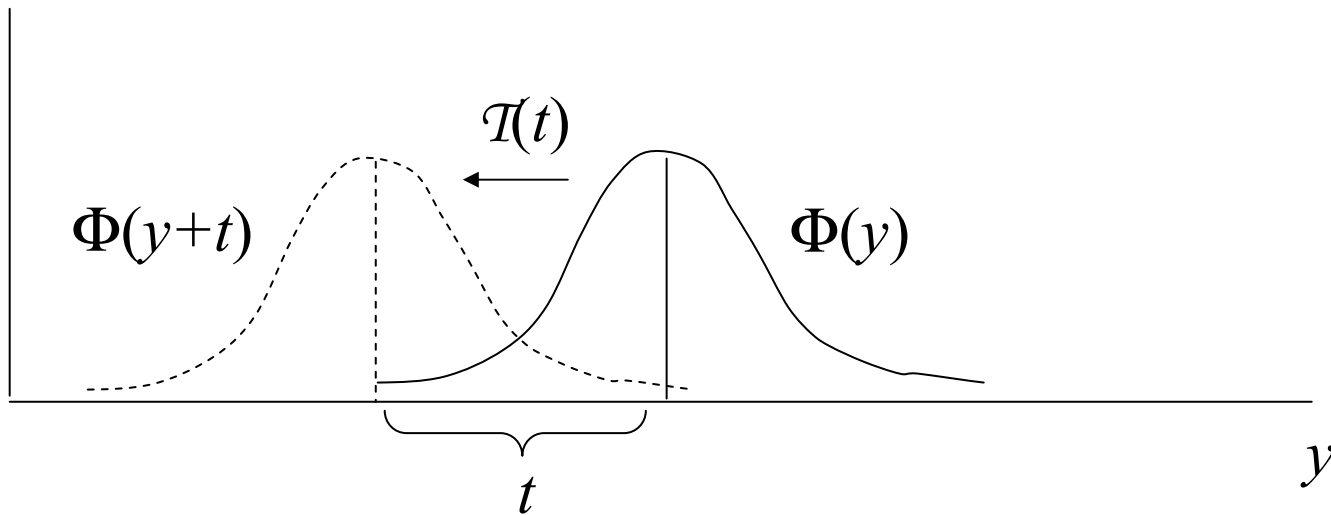
Translation Operator

Operator $\mathcal{T}(\mathbf{t}) : \mathbb{F}(\Omega) \rightarrow \mathbb{F}(\Omega')$, $\Omega' \subset \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is called *translation* operator corresponding to *translation* vector \mathbf{t} , if

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t}), \quad (\mathbf{y} \in \Omega, \quad \mathbf{y} + \mathbf{t} \in \Omega').$$



Example of Translation Operator



R|R-reexpansion

Let $\mathbf{y} - \mathbf{x}_* \in \Omega_r(\mathbf{x}_*) \subset \mathbb{R}^d$, $\Omega_r(\mathbf{x}_*) : |\mathbf{y} - \mathbf{x}_*| < r$, and $\{R_n(\mathbf{y} - \mathbf{x}_*)\}$ be a regular basis in $C(\Omega)$. Let $\mathbf{y} - \mathbf{x}_* + \mathbf{t} \in \Omega_r(\mathbf{x}_*)$ and

$$R_n(\mathbf{y} - \mathbf{x}_* + \mathbf{t}) = \sum_{l=0}^{\infty} (R|R)_{ln}(\mathbf{t}) R_l(\mathbf{y} - \mathbf{x}_*).$$

Coefficients $(R|R)_{ln}(\mathbf{t})$ are called *R|R - reexpansion coefficients* (regular-to-regular), and infinite matrix

$$(R|R)(\mathbf{t}) = \begin{pmatrix} (R|R)_{00} & (R|R)_{01} & \dots \\ (R|R)_{10} & (R|R)_{11} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

is called *R|R - reexpansion matrix*.

Example of R|R-reexpansion

$$R_m(x) = x^m,$$

$$\begin{aligned} R_m(x+t) &= (x+t)^m = x^m + \binom{m}{1}x^{m-1}t + \dots + \binom{m}{m-1}xt^{m-1} + t^m \\ &= \sum_{l=0}^m \binom{m}{l} t^l x^{m-l} = \sum_{l=0}^m \binom{m}{l} t^{m-l} x^l = \sum_{l=0}^m \binom{m}{l} t^{m-l} R_l(x), \end{aligned}$$

$$(R|R)_{lm}(t) = \begin{cases} \binom{m}{l} t^{m-l}, & l \leq m, \\ 0, & l > m. \end{cases}$$

R|R-translation operator

Translation operator $\mathcal{T}(\mathbf{t})$ which is represented in regular basis $\{R_n(\mathbf{y} - \mathbf{x}_*)\}$ by the $R|R$ – *reexpansion matrix* is called $\mathcal{R}|\mathcal{R}$ -translation operator.

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t})$$

$$(\mathcal{R}|\mathcal{R})(\mathbf{t}) = \mathcal{T}(\mathbf{t}).$$

Why the same operator named differently?

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t})$$

The first letter shows
the basis for $\Phi(\mathbf{y})$

$$\mathcal{T}(\mathbf{t}) = \begin{cases} (\mathcal{R}|\mathcal{R})(\mathbf{t}) \\ (\mathcal{S}|\mathcal{S})(\mathbf{t}) \\ (\mathcal{S}|\mathcal{R})(\mathbf{t}) \\ (\mathcal{R}|\mathcal{S})(\mathbf{t}) \end{cases}$$

The second letter
shows the basis
for $\Phi(\mathbf{y} + \mathbf{t})$

Needed only to show the expansion basis
(for operator representation)

Matrix representation of $\mathcal{R}|\mathcal{R}$ -translation operator

Consider
$$\Phi(\mathbf{y}) = \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) R_n(\mathbf{y} - \mathbf{x}_*).$$

$$\Phi(\mathbf{y} + \mathbf{t}) = (\mathcal{R}|\mathcal{R})(\mathbf{t})[\Phi(\mathbf{y})] = \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) (\mathcal{R}|\mathcal{R})(\mathbf{t})[R_n(\mathbf{y} - \mathbf{x}_*)]$$

$$= \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) R_n(\mathbf{y} - \mathbf{x}_* + \mathbf{t})$$

$$= \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) \sum_{l=0}^{\infty} (\mathcal{R}|\mathcal{R})_{ln}(\mathbf{t}) R_l(\mathbf{y} - \mathbf{x}_*)$$

$$= \sum_{l=0}^{\infty} \left[\sum_{n=0}^{\infty} (\mathcal{R}|\mathcal{R})_{ln}(\mathbf{t}) A_n(\mathbf{x}_*) \right] R_l(\mathbf{y} - \mathbf{x}_*)$$

Coefficients of shifted function

$$= \sum_{l=0}^{\infty} \tilde{A}_l(\mathbf{x}_*, \mathbf{t}) R_l(\mathbf{y} - \mathbf{x}_*),$$

Coefficients of original function

$$\tilde{A}_l(\mathbf{x}_*, \mathbf{t}) = \sum_{n=0}^{\infty} (\mathcal{R}|\mathcal{R})_{ln}(\mathbf{t}) A_n(\mathbf{x}_*), \quad \tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) = (\mathcal{R}|\mathcal{R})(\mathbf{t}) \mathbf{A}(\mathbf{x}_*).$$

Reexpansion of the same function over shifted basis

Compact notation:

$$\Phi(\mathbf{y}) = \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) R_n(\mathbf{y} - \mathbf{x}_*) = \mathbf{A}(\mathbf{x}_*) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_*),$$

$$\Phi(\mathbf{y} + \mathbf{t}) = \sum_{l=0}^{\infty} \tilde{A}_l(\mathbf{x}_*, \mathbf{t}) R_l(\mathbf{y} - \mathbf{x}_*) = \tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_*)$$

We have:

$$\begin{aligned} \Phi(\mathbf{y}) &= \Phi((\mathbf{y} - \mathbf{t}) + \mathbf{t}) = \tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) \circ \mathbf{R}((\mathbf{y} - \mathbf{t}) - \mathbf{x}_*) \\ &= \tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_* - \mathbf{t}). \end{aligned}$$

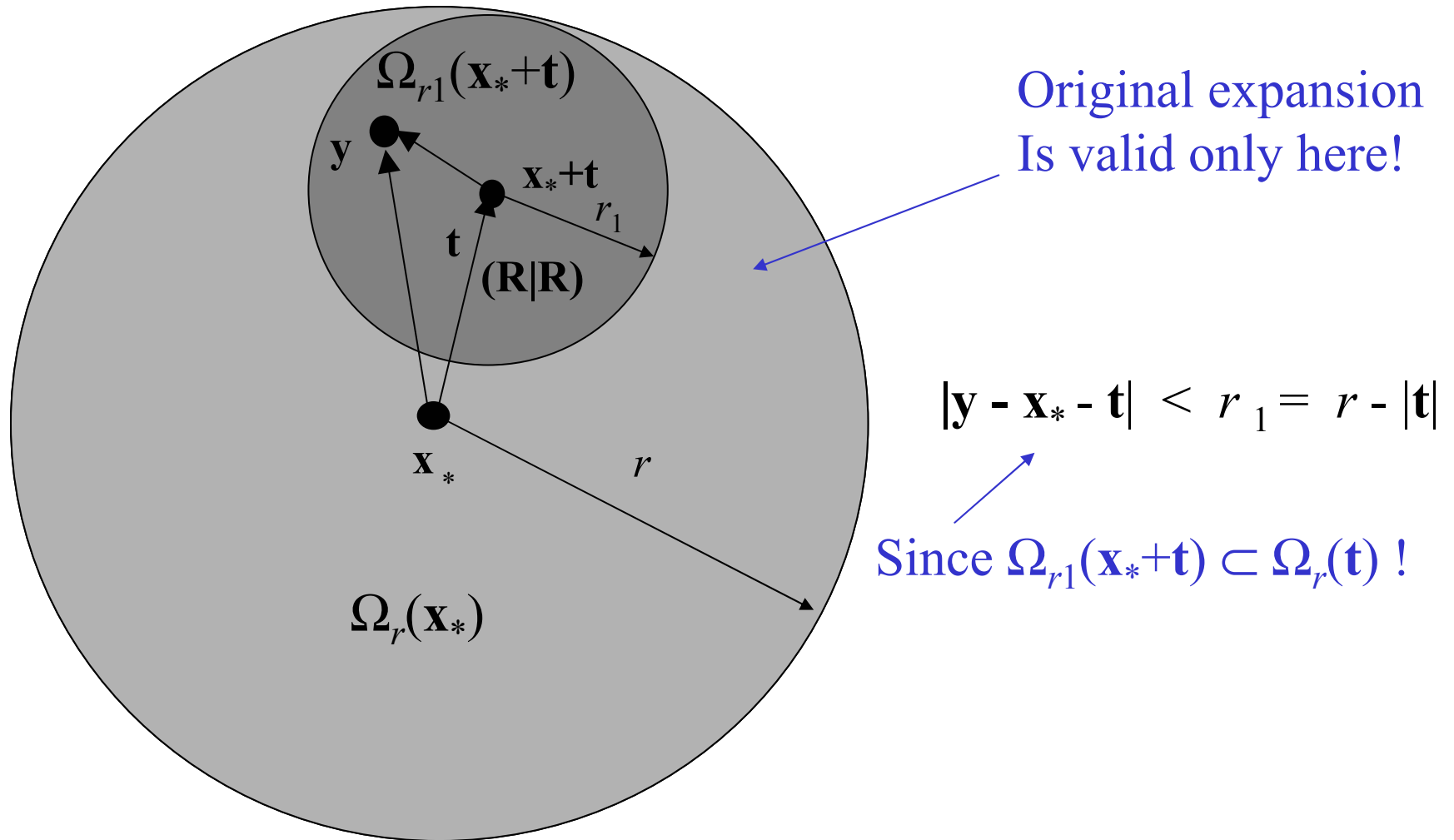
Also

$$\Phi(\mathbf{y}) = \mathbf{A}(\mathbf{x}_*) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_*) = \mathbf{A}(\mathbf{x}_* + \mathbf{t}) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_* - \mathbf{t}),$$

so

$$\mathbf{A}(\mathbf{x}_* + \mathbf{t}) = \tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) = (\mathbf{R}|\mathbf{R})(\mathbf{t})\mathbf{A}(\mathbf{x}_*).$$

R|R-reexpansion of the same function over shifted basis (2)



Example of power series reexpansion

$$R_m(x) = x^m,$$

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} A_m(x_{*1}, x_i) R_m(y - x_{*1}) = \sum_{m=0}^{\infty} A_m(x_{*2}, x_i) R_m(y - x_{*2}),$$

$$\mathbf{A}(x_{*2}, x_i) = (\mathbf{R}|\mathbf{R})(x_{*2} - x_{*1}) \cdot \mathbf{A}(x_{*1}, x_i).$$

$$\begin{pmatrix} A_0(x_{*2}, x_i) \\ A_1(x_{*2}, x_i) \\ A_2(x_{*2}, x_i) \\ \dots \end{pmatrix} = \begin{pmatrix} 1 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} (x_{*2} - x_{*1}) & \begin{pmatrix} 2 \\ 0 \end{pmatrix} (x_{*2} - x_{*1})^2 & \dots \\ 0 & 1 & \begin{pmatrix} 2 \\ 1 \end{pmatrix} (x_{*2} - x_{*1}) & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} A_0(x_{*1}, x_i) \\ A_1(x_{*1}, x_i) \\ A_2(x_{*1}, x_i) \\ \dots \end{pmatrix}$$

Example of power series reexpansion (2). Relation to Taylor series.

Let's check this for Taylor series, when expansion coefficients are

$$A_m(x_{*1}, x_i) = \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_i)}{\partial x_{*1}^m}$$

For A_0 this yields Taylor series again!

Check for A_l

$$\Phi(x_{*2}, x_i) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_i)}{\partial x_{*1}^m} (x_{*2} - x_{*1})^m,$$

$$\frac{1}{l!} \frac{\partial^l \Phi(x_{*2}, x_i)}{\partial x_{*2}^l} = \sum_{m=l}^{\infty} \binom{m}{l} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_i)}{\partial x_{*1}^m} (x_{*2} - x_{*1})^{m-l}$$

$$= \sum_{m=l}^{\infty} \frac{m!}{l!(m-l)!} \frac{1}{m!} \frac{\partial^m \Phi(x_{*1}, x_i)}{\partial x_{*1}^m} (x_{*2} - x_{*1})^{m-l}$$

$$= \frac{1}{l!} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{k+l} \Phi(x_{*1}, x_i)}{\partial x_{*1}^{k+l}} (x_{*2} - x_{*1})^k,$$

$$\frac{\partial^l \Phi(x_{*2}, x_i)}{\partial x_{*2}^l} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{k+l} \Phi(x_{*1}, x_i)}{\partial x_{*1}^{k+l}} (x_{*2} - x_{*1})^k.$$

For A_l we
obtained Taylor
series for the l -th
derivative! Wow!

S|S-reexpansion

Let $\mathbf{y} - \mathbf{x}_* \in \Omega_r(\mathbf{x}_*) \subset \mathbb{R}^d$, $\Omega_r(\mathbf{x}_*) : |\mathbf{y} - \mathbf{x}_*| > r$, and $\{S_n(\mathbf{y} - \mathbf{x}_*)\}$ be a singular basis in $C(\Omega)$. Let $\mathbf{y} - \mathbf{x}_* + \mathbf{t} \in \Omega_r(\mathbf{x}_*)$ and

$$S_n(\mathbf{y} - \mathbf{x}_* + \mathbf{t}) = \sum_{l=0}^{\infty} (S|S)_{ln}(\mathbf{t}) S_l(\mathbf{y} - \mathbf{x}_*).$$

Coefficients $(S|S)_{ln}(\mathbf{t})$ are called *S|S-reexpansion coefficients* (singular-to-singular), and infinite matrix

$$(S|S)(\mathbf{t}) = \begin{pmatrix} (S|S)_{00} & (S|S)_{01} & \dots \\ (S|S)_{10} & (S|S)_{11} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

is called *S|S-reexpansion matrix*.

S|S-translation operator

Translation operator $T(\mathbf{t})$ which is represented in singular basis $\{S_n(\mathbf{y} - \mathbf{x}_*)\}$ by the $\mathcal{S}|\mathcal{S}$ -*reexpansion matrix* is called $\mathcal{S}|\mathcal{S}$ -translation operator.

$$T(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t})$$

$$(\mathcal{S}|\mathcal{S})(\mathbf{t}) = T(\mathbf{t}).$$

S|S and R|R-translation operators are very similar,

(actually, this is just two representations of
the same translation operator in different domains and bases)

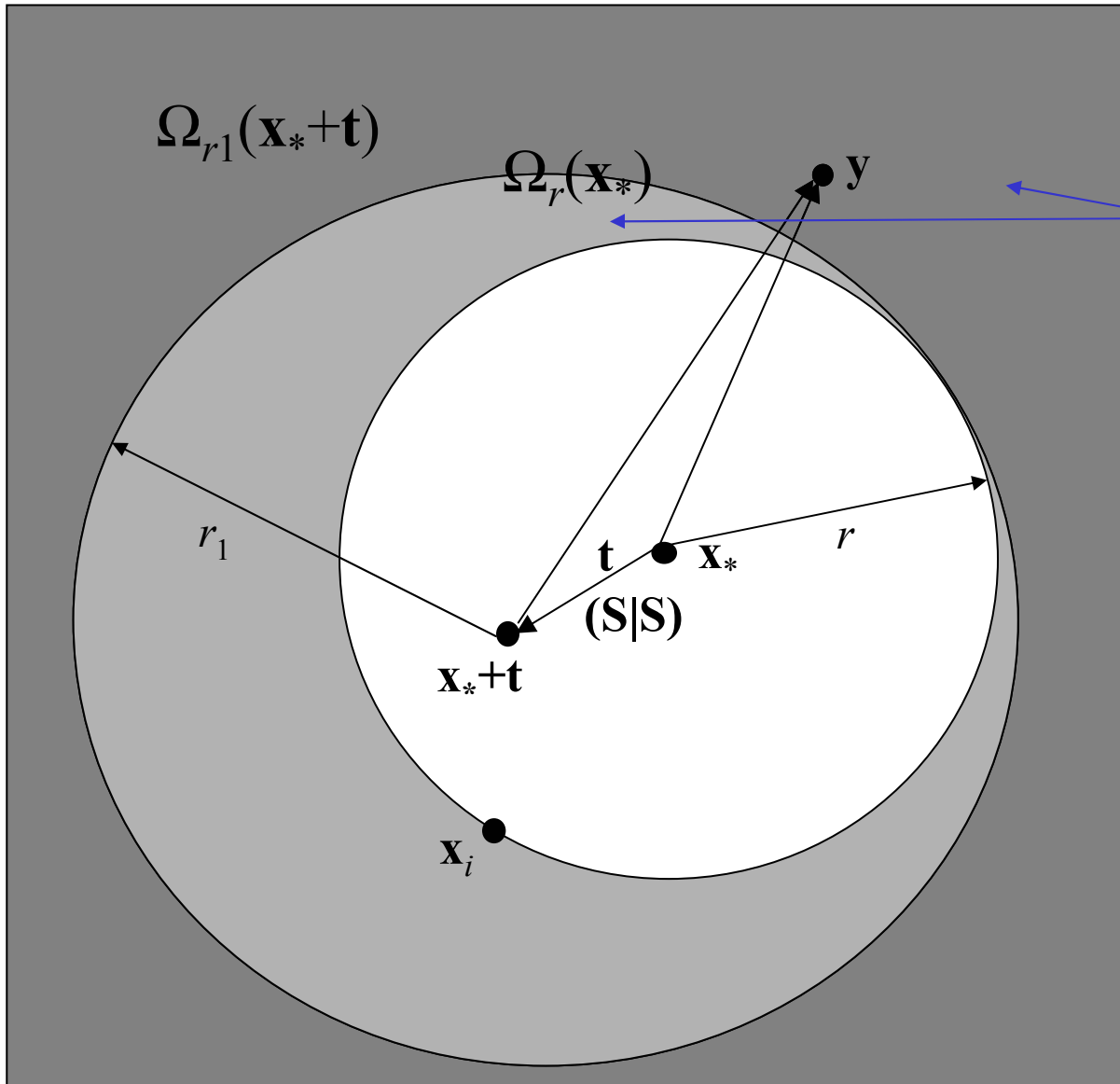
$$\Phi(\mathbf{y}) = \mathbf{B}(\mathbf{x}_*) \circ \mathbf{S}(\mathbf{y} - \mathbf{x}_*),$$

$$\Phi(\mathbf{y} + \mathbf{t}) = \tilde{\mathbf{B}}(\mathbf{x}_*, \mathbf{t}) \circ \mathbf{S}(\mathbf{y} - \mathbf{x}_*)$$

$$\Phi(\mathbf{y}) = \tilde{\mathbf{B}}(\mathbf{x}_*, \mathbf{t}) \circ \mathbf{S}(\mathbf{y} - \mathbf{x}_* - \mathbf{t}).$$

$$\tilde{\mathbf{B}}(\mathbf{x}_*, \mathbf{t}) = (\mathbf{S}|\mathbf{S})(\mathbf{t})\mathbf{B}(\mathbf{x}_*) = \mathbf{B}(\mathbf{x}_* + \mathbf{t}).$$

But picture is different...



Original expansion
Is valid only here!

$$|\mathbf{y} - \mathbf{x}_* - \mathbf{t}| > r_1 = r + |\mathbf{t}|$$

Since

$$\Omega_{r_1}(\mathbf{x}_* + \mathbf{t}) \subset \Omega_r(\mathbf{t}) !$$

Also

$$|\mathbf{x}_i - \mathbf{x}_*| < r$$

singular point !

S|R-reexpansion

Let $\mathbf{y} - \mathbf{x}_* \in \Omega_r(\mathbf{x}_*) \subset \mathbb{R}^d$, $\Omega_r(\mathbf{x}_*) : |\mathbf{y} - \mathbf{x}_*| < r$, and $\{R_n(\mathbf{y} - \mathbf{x}_*)\}$ be a regular basis in $C(\Omega_r(\mathbf{x}_*))$. Let also $\Omega_{r_1}(\mathbf{x}_* - \mathbf{t}) : |\mathbf{y} - \mathbf{x}_* + \mathbf{t}| > R > r$, and $\{S_n(\mathbf{y} - \mathbf{x}_* + \mathbf{t})\}$ be a singular basis in $C(\Omega_r(\mathbf{x}_*))$, then

$$S_n(\mathbf{y} - \mathbf{x}_* + \mathbf{t}) = \sum_{l=0}^{\infty} (S|R)_{ln}(\mathbf{t}) R_l(\mathbf{y} - \mathbf{x}_*).$$

Coefficients $(S|R)_{ln}(\mathbf{t})$ are called *S|R-reexpansion coefficients* (singular-to-regular), and infinite matrix

$$(S|R)(\mathbf{t}) = \begin{pmatrix} (S|R)_{00} & (S|R)_{01} & \dots \\ (S|R)_{10} & (S|R)_{11} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

is called *S|R-reexpansion matrix*.

Does R|S reexpansion exist?

- Theoretically yes (in some cases, e.g. analytical continuation);
- In practice, since the domain of S-expansion is larger than the domain of R-expansion, this either not useful (due to error bounds), or can be avoided in algorithms;
- We will not use R|S-reexpansions in the FMM algorithms.

S|R-translation operator

Translation operator $\mathcal{T}(\mathbf{t})$ which is represented in singular basis by the $S|R$ – *reexpansion matrix* is called $\mathcal{S}|\mathcal{R}$ -translation operator if the basis of expansion is changed with the translation operation from singular $\{S_n(\mathbf{y} - \mathbf{x}_*)\}$ to regular $\{R_n(\mathbf{y} - \mathbf{x}_* + \mathbf{t})\}$

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t})$$

$$(\mathcal{S}|\mathcal{R})(\mathbf{t}) = \mathcal{T}(\mathbf{t}).$$

S|R-operator has almost the same
properties as S|S and R|R

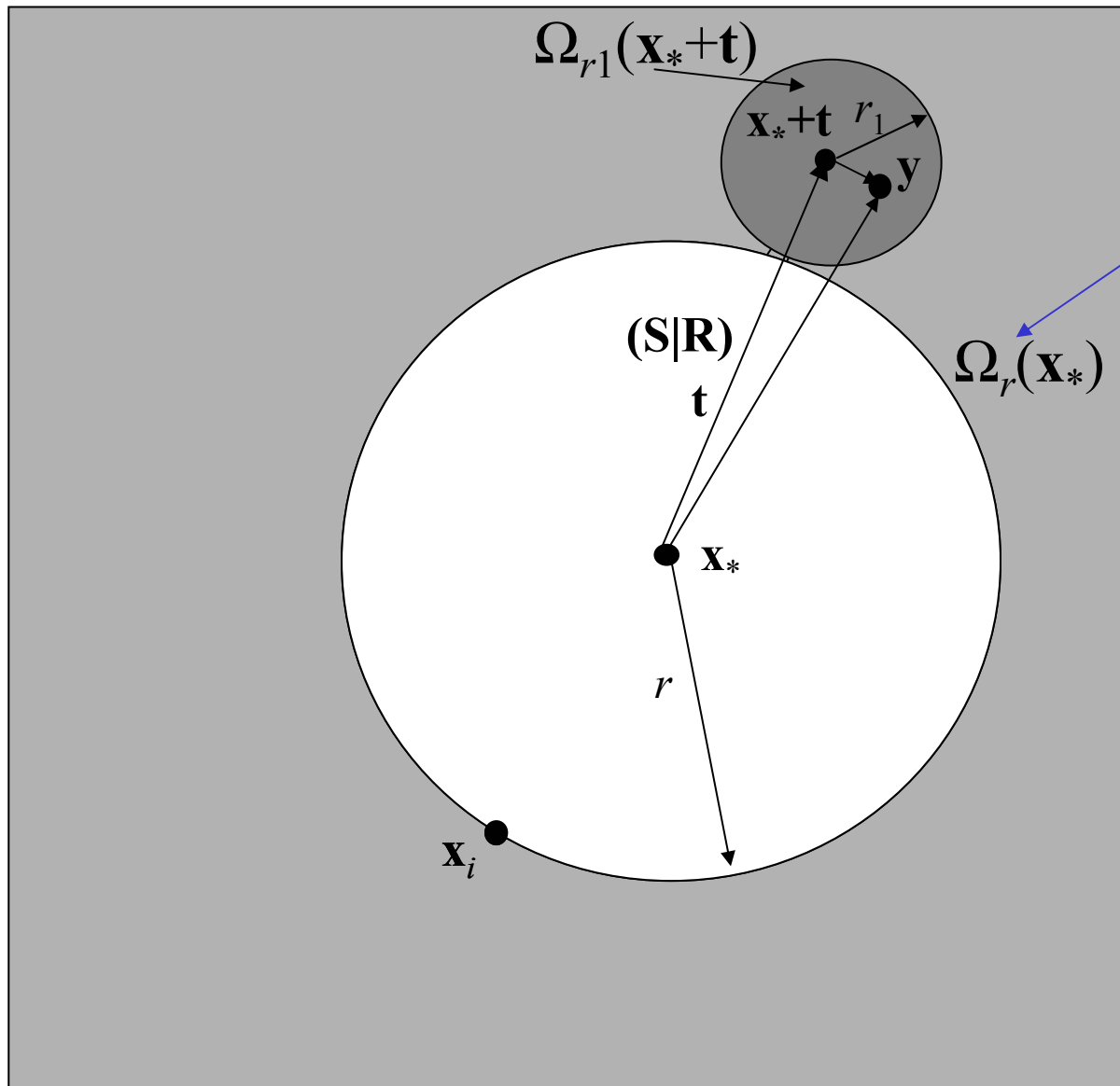
(t cannot be zero)

$$\begin{aligned}\Phi(\mathbf{y}) &= \mathbf{B}(\mathbf{x}_*) \circ \mathbf{S}(\mathbf{y} - \mathbf{x}_*), \\ \Phi(\mathbf{y} + \mathbf{t}) &= \tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_*)\end{aligned}$$

$$\Phi(\mathbf{y}) = \tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_* - \mathbf{t}).$$

$$\tilde{\mathbf{A}}(\mathbf{x}_*, \mathbf{t}) = (\mathbf{S|R})(\mathbf{t})\mathbf{B}(\mathbf{x}_*).$$

Picture is different...



Original expansion
Is valid only here!

$$|\mathbf{y} - \mathbf{x}_* - \mathbf{t}| < r_1 = |\mathbf{t}| - r$$

Since

$$\Omega_{r_1}(\mathbf{x}_* + \mathbf{t}) \subset \Omega_r(\mathbf{t}) !$$

Also

$$|\mathbf{x}_i - \mathbf{x}_*| < r$$

singular point !

Properties of the translation operator

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t})$$

● $\mathcal{T}(\mathbf{0}) = \mathcal{I}$ (identity operator). Proof:

$$\mathcal{T}(\mathbf{0})[\Phi(\mathbf{y})] = \Phi(\mathbf{y}).$$

● $\mathcal{T}(\mathbf{t}_1 + \mathbf{t}_2) = \mathcal{T}(\mathbf{t}_1) \circ \mathcal{T}(\mathbf{t}_2) = \mathcal{T}(\mathbf{t}_2) \circ \mathcal{T}(\mathbf{t}_1)$. Proof:

$$\mathcal{T}(\mathbf{t}_1) \circ \mathcal{T}(\mathbf{t}_2)[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t}_2 + \mathbf{t}_1) = \mathcal{T}(\mathbf{t}_2 + \mathbf{t}_1)[\Phi(\mathbf{y})] = \mathcal{T}(\mathbf{t}_1 + \mathbf{t}_2)[\Phi(\mathbf{y})].$$

● (corollary 1): $\mathcal{T}^{-1}(\mathbf{t}) = \mathcal{T}(-\mathbf{t})$. Proof:

$$\mathcal{I} = \mathcal{T}(\mathbf{0}) = \mathcal{T}(\mathbf{t} - \mathbf{t}) = \mathcal{T}(\mathbf{t}) \circ \mathcal{T}(-\mathbf{t}).$$

● (corollary 2): $\mathcal{T}^n(\mathbf{t}) = \mathcal{T}(n\mathbf{t})$. Proof (use induction):

$$\mathcal{T}(n\mathbf{t}) = \mathcal{T}((n-1)\mathbf{t}) \circ \mathcal{T}(\mathbf{t}) = \mathcal{T}^{n-1}(\mathbf{t}) \circ \mathcal{T}(\mathbf{t}) = \mathcal{T}^n(\mathbf{t}).$$

Spectrum of the translation operator

eigen value *eigen function*

$$\mathcal{T}(\mathbf{t})[\Psi(\mathbf{y})] = \lambda\Psi(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^d.$$

Any function of type

$$\forall \mathbf{a} \in \mathbb{R}^d, \quad \Psi(\mathbf{y}) = e^{\mathbf{a}\cdot\mathbf{y}}, \quad \lambda = e^{\mathbf{a}\cdot\mathbf{t}}.$$

Check:

$$\mathcal{T}(\mathbf{t})[\Psi(\mathbf{y})] = \Psi(\mathbf{y} + \mathbf{t}) = e^{\mathbf{a}\cdot(\mathbf{y}+\mathbf{t})} = e^{\mathbf{a}\cdot\mathbf{t}}e^{\mathbf{a}\cdot\mathbf{y}} = \lambda\Psi(\mathbf{y}).$$

Relation to differential operator:

$$\frac{d\Phi(\mathbf{y})}{ds} = \lim_{|\mathbf{t}|\rightarrow 0} \frac{\Phi(\mathbf{y} + \mathbf{t}) - \Phi(\mathbf{y})}{|\mathbf{t}|} = \lim_{|\mathbf{t}|\rightarrow 0} \frac{\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] - \Phi(\mathbf{y})}{|\mathbf{t}|} = \lim_{|\mathbf{t}|\rightarrow 0} \frac{\mathcal{T}(\mathbf{t}) - \mathcal{I}}{|\mathbf{t}|}[\Phi(\mathbf{y})], \quad \mathbf{s} = \frac{\mathbf{t}}{|\mathbf{t}|}.$$

derivative in direction s