

MAIT 627 Fast Multipole Methods

Lecture 6

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Outline

- Representation of functions in the space of coefficients
- Matrix representation of operators
- Truncation and truncated operators
- Translation operator
- Reexpansion coefficients
- RIR and SIS translation operators
- Examples
- SIR and RIS translation operators
- Properties of translation operators

Why do we need represent functions in different spaces?

- Functions should be efficiently summed up;
- Sums of functions should be compressed;
- Error bounds should be established;
- Functions should be translated and expanded over different bases;
- For computations we need discrete and finite function representations.
- Some functions measured experimentally or approximated by splines, and there is no explicit analytical representation in the whole space.

Linear Spaces

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{U}$$

- 1). $\mathbf{a} + \mathbf{b} \in \mathcal{U}$;
- 2). $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$;
- 3). $\exists \mathbf{0}, \mathbf{a} + \mathbf{0} = \mathbf{a}, \mathbf{a} + (-\mathbf{a}) = \mathbf{a} - \mathbf{a} = \mathbf{0}$;
- 4). $\forall \alpha \in \mathbb{C}, \alpha \mathbf{a} \in \mathcal{U}$;
- 5). $\forall \alpha, \beta \in \mathbb{C}, (\alpha\beta)\mathbf{a} = \alpha(\beta)\mathbf{a}, 1\mathbf{a} = \mathbf{a}, \alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}, (\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$.

Linear Operators

Linear Spaces

$$\psi \in \mathbb{F}(\Omega), \quad \psi' \in \mathbb{F}(\Omega'), \quad \Omega, \Omega' \subset \mathbb{R}^d,$$

Operator

$$\psi' = \mathcal{A}[\psi],$$

Linear Operator

$$\mathcal{A}[\alpha\psi_1 + \beta\psi_2] = \alpha\mathcal{A}[\psi_1] + \beta\mathcal{A}[\psi_2], \quad \alpha, \beta \in \mathbb{C}.$$

An example of linear operator: Differential Operator.

Representation of Functions and

$$\psi \in \mathbb{F}(\Omega), \quad \psi' \in \mathbb{F}(\Omega'), \quad \Omega, \Omega' \subset \mathbb{R}^d.$$

Bases $\longrightarrow F_n \in \mathbb{F}(\Omega), \quad F'_n \in \mathbb{F}(\Omega'),$

$$\psi = \sum_n c_n F_n, \quad \psi' = \sum_n c'_n F'_n,$$

$$\mathcal{A}[F_n] = \sum_{n'} (F|F')_{n'n} F'_{n'}$$

Reexpansion Coefficients

$$\mathcal{A}[\psi] = \mathcal{A}\left[\sum_n c_n F_n\right] = \sum_n c_n \mathcal{A}[F_n] =$$

$$= \sum_n c_n \sum_{n'} (F|F')_{n'n} F'_{n'} = \sum_{n'} \left[\sum_n (F|F')_{n'n} c_n\right] F'_{n'} = \sum_{n'} c'_n F'_{n'} = \psi'$$

$$c'_{n'} = \sum_n (F|F')_{n'n} c_n.$$

Matrix Representation of operator A

Function Representation in the Space of Coefficients

Let $\mathbb{F}(\Omega) \subset C(\Omega)$, $\Omega \subset \mathbb{R}^d$, be a normed space of continuous functions with norm

$$\|\Phi(\mathbf{y})\| = \max_{\mathbf{y} \in \Omega} |\Phi(\mathbf{y})|.$$

Let also $\{F_n(\mathbf{y})\}$ be a complete basis in $\mathbb{F}(\Omega)$, so

$$\Phi(\mathbf{y}) = \sum_{n=0}^{\infty} A_n F_n(\mathbf{y}), \quad \mathbf{y} \in \Omega \subset \mathbb{R}^d, \quad \Phi(\mathbf{y}), F_n(\mathbf{y}) \in \mathbb{F}(\Omega),$$

absolutely and uniformly converges in $\Omega \subset \mathbb{R}^d$. This means that

$$\forall \epsilon > 0, \quad \exists p(\epsilon), \quad |\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})| < \epsilon, \quad \forall \mathbf{y} \in \Omega,$$

$$\forall \epsilon > 0, \quad \exists p(\epsilon), \quad \sum_{n=p}^{\infty} |A_n F_n(\mathbf{y})| < \epsilon, \quad \forall \mathbf{y} \in \Omega,$$

$$\Phi^p(\mathbf{y}) = \sum_{n=0}^{p-1} A_n F_n(\mathbf{y}).$$

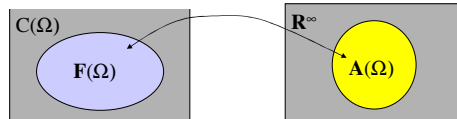
Function Representation in the Space of Coefficients (2)

Expansion coefficients can be stacked in an infinite vector

$$\mathbf{A} = \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_n \\ \dots \end{pmatrix}.$$

Let us denote $\mathbb{A}(\Omega)$ a subset of \mathbb{R}^{∞} which is an image of $\mathbb{F}(\Omega)$. For any $\mathbf{A} \in \mathbb{A}(\Omega)$ we request that there exists one-to-one mapping

$$\Phi(\mathbf{y}) \cong \mathbf{A}, \quad \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \quad \mathbf{A} \in \mathbb{A}(\Omega) \subset \mathbb{R}^{\infty}.$$



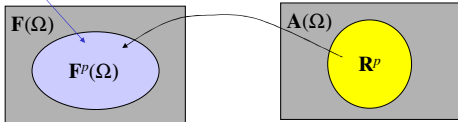
p-Truncated Vectors

$$\forall \mathbf{A} \in \mathbb{R}^p, \exists \Phi^p(\mathbf{y}) = \sum_{n=0}^{p-1} A_n F_n(\mathbf{y}) \in \mathbb{F}^p(\Omega) \subset \mathbb{F}(\Omega).$$

$\mathbb{F}^p(\Omega)$ is dense in $\mathbb{F}(\Omega)$:

$$\forall \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \exists p, \Phi^p(\mathbf{y}) \in \mathbb{F}^p(\Omega), \|\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})\| = \sup_{\mathbf{y} \in \Omega} |\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})| < \epsilon.$$

Dense in $\mathbb{F}(\Omega)$



Matrix Representation of Linear Operators

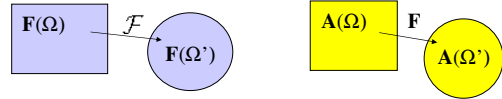
Let $\Omega' \subset \Omega$ and \mathcal{F} is a mapping of $\mathbb{F}(\Omega)$ to $\mathbb{F}(\Omega')$. Such mapping can be considered as action of operator \mathcal{F} on Φ :

$$\mathcal{F}[\Phi(\mathbf{y})] = \widehat{\Phi}(\mathbf{y}'), \quad \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \quad \widehat{\Phi}(\mathbf{y}') \in \mathbb{F}(\Omega') \subset \mathbb{F}(\Omega)$$

Respectively, operator \mathcal{F} generates operator \mathbf{F} that maps the space of expansion coefficients $\mathbb{A}(\Omega) \rightarrow \mathbb{A}(\Omega')$, which can be considered as *representation* of the operator \mathcal{F} in the space of expansion coefficients:

$$\mathbf{F}\mathbf{A} = \tilde{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{A}(\Omega), \quad \tilde{\mathbf{A}} \in \mathbb{A}(\Omega') \subset \mathbb{A}(\Omega).$$

Inversly, if we introduce any transform of expansion coefficients $\mathbf{F}\mathbf{A} = \tilde{\mathbf{A}}$ which provides uniform convergence of function $\widehat{\Phi}(\mathbf{y}')$ corresponding to these coefficients in $\Omega' \subset \Omega$ then such transform can be treated as operator \mathcal{F} that convert one function from $\mathbb{F}(\Omega)$ to another.



Representation of a Linear Operator

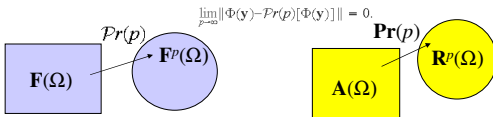
p-Truncation (Projection) Operator

$$\text{Pr}(p)\mathbf{A} = \tilde{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{A}(\Omega), \quad \tilde{\mathbf{A}} \in \mathbb{A}^p(\Omega).$$

$$\mathbf{A} = \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_{p-1} \\ A_p \\ A_{p+1} \\ \dots \end{pmatrix} \rightarrow \tilde{\mathbf{A}} = \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_{p-1} \\ 0 \\ 0 \\ \dots \end{pmatrix}, \quad \tilde{\mathbf{A}} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \mathbf{A}$$

In space $\mathbb{F}(\Omega)$:

$$\text{Pr}(p)[\Phi(\mathbf{y})] = \Phi^p(\mathbf{y}), \quad \Phi(\mathbf{y}) \in \mathbb{F}(\Omega), \quad \Phi^p(\mathbf{y}) \in \mathbb{F}^p(\Omega),$$



Norm of p-Truncation Operator (important for error bounds)

Norm:

$$\|\text{Pr}(p)\| = \frac{\sup_{\mathbf{y} \in \Omega} \|\text{Pr}(p)[\Phi(\mathbf{y})]\|}{\sup_{\mathbf{y} \in \Omega} \|\Phi(\mathbf{y})\|}.$$

Triangle inequality:

$$\|\mathbf{I} - \|\mathbf{I} - \text{Pr}(p)\| \leq \|\text{Pr}(p)\| \leq \|\mathbf{I}\| + \|\mathbf{I} - \text{Pr}(p)\| = 1 + \|\mathbf{I} - \text{Pr}(p)\|$$

$$\forall \epsilon > 0, \exists p, \|\mathbf{I} - \text{Pr}(p)\| < \epsilon,$$

so

$$\forall \epsilon > 0, \exists p, 1 - \epsilon < \|\text{Pr}(p)\| < 1 + \epsilon,$$

p-Truncated Operator

Let $H : F(\Omega) \rightarrow F(\Omega)$ be an operator, that is represented by infinite matrix

$$H = \begin{pmatrix} h_{00} & h_{01} & \dots & h_{0,p-1} & h_{0p} & \dots \\ h_{10} & h_{11} & \dots & h_{1,p-1} & h_{1p} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_{p-1,0} & h_{p-1,1} & \dots & h_{p-1,p-1} & h_{p-1,p} & \dots \\ h_{p0} & h_{p1} & \dots & h_{p-1,p} & h_{pp} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

We call operator $H^{(p)} : F(\Omega) \rightarrow F(\Omega)$, *p-truncated* if it is represented by matrix

$$H^{(p)} = \begin{pmatrix} h_{00} & h_{01} & \dots & h_{0,p-1} & 0 & \dots \\ h_{10} & h_{11} & \dots & h_{1,p-1} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_{p-1,0} & h_{p-1,1} & \dots & h_{p-1,p-1} & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Norm of p-Truncated Operator (important for error bounds)

Theorem: Let $H : F(\Omega) \rightarrow F(\Omega)$, such that $0 < \|H\| < \infty$, and $H^{(p)} : F(\Omega) \rightarrow F(\Omega)$ is the *p-truncated operator* H . Let also $p(\epsilon)$ be such that $1 - \epsilon < \|Pr(p)\| < 1 + \epsilon$. Then

$$(1 - \epsilon)^2 < \|Pr(p)\|^2 = \frac{\|H^{(p)}\|}{\|H\|} = \|Pr(p)\|^2 < (1 + \epsilon)^2,$$

$$\lim_{p \rightarrow \infty} \frac{\|H^{(p)}\|}{\|H\|} = 1.$$

Proof.

A *p-truncated operator* can be represented in the form

$$H^{(p)} = Pr(p)HPr(p)$$

(check!)

So the norm of $H^{(p)}$ is

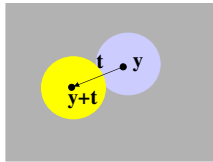
$$\|H^{(p)}\| = \|Pr(p)\| \|H\| \|Pr(p)\| = \|H\| \|Pr(p)\|^2.$$

End of Proof.

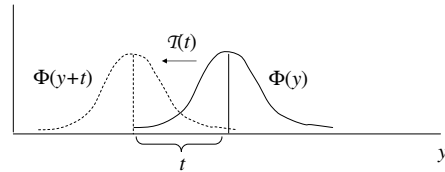
Translation Operator

Operator $\mathcal{T}(t) : F(\Omega) \rightarrow F(\Omega')$, $\Omega' \subset \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is called *translation operator* corresponding to *translation vector* t , if

$$\mathcal{T}(t)[\Phi(y)] = \Phi(y+t), \quad (y \in \Omega, \quad y+t \in \Omega').$$



Example of Translation Operator



R|R-reexpansion

Let $y - x_* \in \Omega_r(x_*) \subset \mathbb{R}^d$, $\Omega_r(x_*) : |y - x_*| < r$, and $\{R_n(y - x_*)\}$ be a regular basis in $C(\Omega)$. Let $y - x_* + t \in \Omega_r(x_*)$ and

$$R_n(y - x_* + t) = \sum_{j=0}^{\infty} (R|R)_{jn}(t) R_j(y - x_*).$$

Coefficients $(R|R)_{jn}(t)$ are called *R|R-reexpansion coefficients* (regular-to-regular), and infinite matrix

$$(R|R)(t) = \begin{pmatrix} (R|R)_{00} & (R|R)_{01} & \dots \\ (R|R)_{10} & (R|R)_{11} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

is called *R|R-reexpansion matrix*.

Example of R|R-reexpansion

$$R_m(x) = x^m,$$

$$\begin{aligned} R_m(x+t) &= (x+t)^m = x^m + \binom{m}{1} x^{m-1} t + \dots + \binom{m}{m-1} x t^{m-1} + t^m \\ &= \sum_{l=0}^m \binom{m}{l} t^l x^{m-l} = \sum_{l=0}^m \binom{m}{l} t^{m-l} x^l = \sum_{l=0}^m \binom{m}{l} t^{m-l} R_l(x), \end{aligned}$$

$$(R|R)_{ln}(t) = \begin{cases} \binom{m}{l} t^{m-l}, & l \leq m, \\ 0, & l > m. \end{cases}$$

R|R-translation operator

Translation operator $\mathcal{T}(t)$ which is represented in regular basis $\{R_n(y - x_*)\}$ by the *R|R-reexpansion matrix* is called *R|R-translation operator*.

$$\mathcal{T}(t)[\Phi(y)] = \Phi(y+t)$$

$$(\mathcal{R}|\mathcal{R})(t) = \mathcal{T}(t).$$

Why the same operator named differently?

$$\mathcal{T}(t)[\Phi(y)] = \Phi(y+t)$$

The first letter shows the basis for $\Phi(y)$

The second letter shows the basis for $\Phi(y+t)$

$$\mathcal{T}(t) = \begin{cases} (\mathcal{R}|\mathcal{R})(t) \\ (\mathcal{S}|\mathcal{S})(t) \\ (\mathcal{S}|\mathcal{R})(t) \\ (\mathcal{R}|\mathcal{S})(t) \end{cases}$$

Needed only to show the expansion basis (for operator representation)

Matrix representation of RIR-translation operator

Consider $\Phi(y) = \sum_{n=0}^{\infty} A_n(x_*) R_n(y - x_*)$.

$$\begin{aligned} \Phi(y+t) &= (\mathcal{R}|\mathcal{R})(t)[\Phi(y)] = \sum_{n=0}^{\infty} A_n(x_*) (\mathcal{R}|\mathcal{R})(t)[R_n(y-x_*)] \\ &= \sum_{n=0}^{\infty} A_n(x_*) R_n(y-x_*+t) \\ &= \sum_{n=0}^{\infty} A_n(x_*) \sum_{l=0}^{\infty} (\mathcal{R}|\mathcal{R})_{ln}(t) R_l(y-x_*) \\ &= \sum_{l=0}^{\infty} \left[\sum_{n=0}^{\infty} (\mathcal{R}|\mathcal{R})_{ln}(t) A_n(x_*) \right] R_l(y-x_*) \quad \text{Coefficients of shifted function} \\ &= \sum_{l=0}^{\infty} \tilde{A}_l(x_*, t) R_l(y-x_*), \end{aligned}$$

Coefficients of original function

$$\tilde{A}_l(x_*, t) = \sum_{n=0}^{\infty} (\mathcal{R}|\mathcal{R})_{ln}(t) A_n(x_*), \quad \tilde{A}(x_*, t) = (\mathcal{R}|\mathcal{R})(t) \tilde{A}(x_*).$$

Reexpansion of the same function over shifted basis

Compact notation:

$$\begin{aligned} \Phi(y) &= \sum_{n=0}^{\infty} A_n(x_*) R_n(y-x_*) = A(x_*) \circ R(y-x_*), \\ \Phi(y+t) &= \sum_{l=0}^{\infty} \tilde{A}_l(x_*, t) R_l(y-x_*) = \tilde{A}(x_*, t) \circ R(y-x_*) \end{aligned}$$

We have:

$$\begin{aligned} \Phi(y) &= \Phi((y-t)+t) = \tilde{A}(x_*, t) \circ R((y-t)-x_*) \\ &= \tilde{A}(x_*, t) \circ R(y-x_*-t). \end{aligned}$$

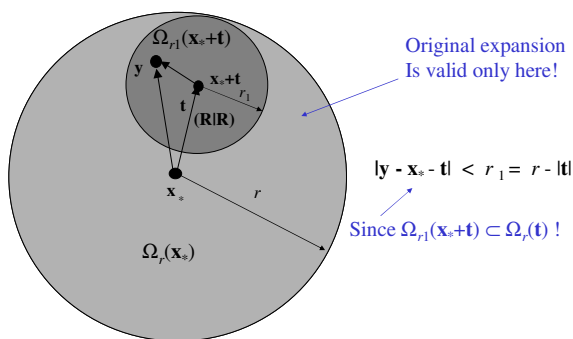
Also

$$\Phi(y) = A(x_*) \circ R(y-x_*) = A(x_*+t) \circ R(y-x_*-t),$$

so

$$A(x_*+t) = \tilde{A}(x_*, t) = (\mathcal{R}|\mathcal{R})(t) A(x_*).$$

RIR-reexpansion of the same function over shifted basis (2)



Example of power series reexpansion

$R_m(x) = x^m$

$$\Phi(y, x_1) = \sum_{m=0}^{\infty} A_m(x_{*1}, x_1) R_m(y-x_{*1}) = \sum_{m=0}^{\infty} A_m(x_{*2}, x_1) R_m(y-x_{*2}),$$

$$A(x_{*2}, x_1) = (\mathcal{R}|\mathcal{R})(x_{*2}-x_{*1}) \cdot A(x_{*1}, x_1).$$

$$\begin{pmatrix} A_0(x_{*2}, x_1) \\ A_1(x_{*2}, x_1) \\ A_2(x_{*2}, x_1) \\ \dots \end{pmatrix} = \begin{pmatrix} 1 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} (x_{*2}-x_{*1}) & \begin{pmatrix} 2 \\ 0 \end{pmatrix} (x_{*2}-x_{*1})^2 & \dots \\ 0 & 1 & \begin{pmatrix} 2 \\ 1 \end{pmatrix} (x_{*2}-x_{*1}) & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} A_0(x_{*1}, x_1) \\ A_1(x_{*1}, x_1) \\ A_2(x_{*1}, x_1) \\ \dots \end{pmatrix}$$

SIS-reexpansion

Let $y - x_* \in \Omega_r(x_*) \subset \mathbb{R}^d$, $\Omega_r(x_*) : |y - x_*| > r$, and $\{S_n(y - x_*)\}$ be a singular basis in $C(\Omega)$. Let $y - x_* + t \in \Omega_r(x_*)$ and

$$S_n(y - x_* + t) = \sum_{j=0}^{\infty} (S|S)_{nj}(t) S_j(y - x_*).$$

Coefficients $(S|S)_{nj}(t)$ are called *S|S-reexpansion coefficients* (singular-to-singular), and infinite matrix

$$(S|S)(t) = \begin{pmatrix} (S|S)_{00} & (S|S)_{01} & \dots \\ (S|S)_{10} & (S|S)_{11} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

is called *S|S-reexpansion matrix*.

SIS-translation operator

Translation operator $\mathcal{T}(t)$ which is represented in singular basis $\{S_n(y - x_*)\}$ by the *S|S-reexpansion matrix* is called *S|S-translation operator*.

$$\mathcal{T}(t)[\Phi(y)] = \Phi(y + t)$$

$$(S|S)(t) = \mathcal{T}(t).$$

SIS and RIR-translation operators are very similar,

(actually, this is just two representations of the same translation operator in different domains and bases)

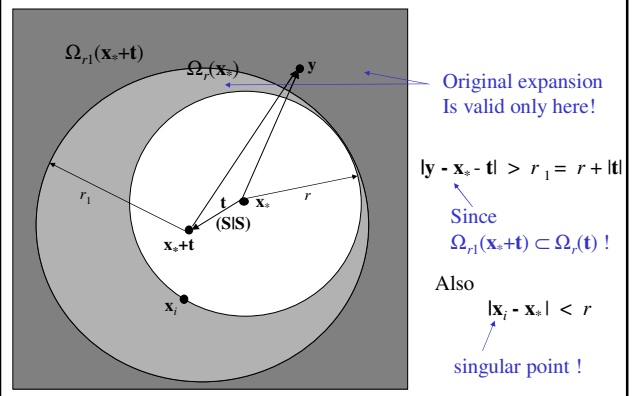
$$\Phi(y) = B(x_*) \circ S(y - x_*),$$

$$\Phi(y + t) = \tilde{B}(x_*, t) \circ S(y - x_*).$$

$$\Phi(y) = \tilde{B}(x_*, t) \circ S(y - x_* - t).$$

$$\tilde{B}(x_*, t) = (S|S)(t)B(x_*) = B(x_* + t).$$

But picture is different...



SIR-reexpansion

Let $\mathbf{y} - \mathbf{x}_* \in \Omega_r(\mathbf{x}_*) \subset \mathbb{R}^d$, $\Omega_r(\mathbf{x}_*) : |\mathbf{y} - \mathbf{x}_*| < r$, and $\langle R_s(\mathbf{y} - \mathbf{x}_*) \rangle$ be a regular basis in $C(\Omega_r(\mathbf{x}_*))$. Let also $\Omega_{r+1}(\mathbf{x}_*, -t) : |\mathbf{y} - \mathbf{x}_* + t| > R > r$, and $\langle S_s(\mathbf{y} - \mathbf{x}_* + t) \rangle$ be a singular basis in $C(\Omega_r(\mathbf{x}_*))$, then

$$S_s(\mathbf{y} - \mathbf{x}_* + t) = \sum_{j=0}^{\infty} (SIR)_{0j}(t) R_j(\mathbf{y} - \mathbf{x}_*).$$

Coefficients $(SIR)_{0j}(t)$ are called *SIR-reexpansion coefficients* (singular-to-regular), and infinite matrix

$$(SIR)(t) = \begin{pmatrix} (SIR)_{00} & (SIR)_{01} & \dots \\ (SIR)_{10} & (SIR)_{11} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

is called *SIR-reexpansion matrix*.

Does RIS reexpansion exist?

- Theoretically yes (in some cases, e.g. analytical continuation);
- In practice, since the domain of S-expansion is larger than the domain of R-expansion, this either not useful (due to error bounds), or can be avoided in algorithms;
- We will not use RIS-reexpansions in the FMM algorithms.

SIR-translation operator

Translation operator $\mathcal{T}(t)$ which is represented in singular basis by the *SIR-reexpansion matrix* is called *SIR-translation operator* if the basis of expansion is changed with the translation operation from singular $\langle S_s(\mathbf{y} - \mathbf{x}_*) \rangle$ to regular $\langle R_s(\mathbf{y} - \mathbf{x}_* + t) \rangle$

$$\mathcal{T}(t)[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + t)$$

$$(SIR)(t) = \mathcal{T}(t).$$

SIR-operator has almost the same properties as SIS and RIR

(t cannot be zero)

$$\Phi(\mathbf{y}) = \mathbf{B}(\mathbf{x}_*) \circ \mathbf{S}(\mathbf{y} - \mathbf{x}_*),$$

$$\Phi(\mathbf{y} + t) = \tilde{\mathbf{A}}(\mathbf{x}_*, t) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_*)$$

$$\Phi(\mathbf{y}) = \tilde{\mathbf{A}}(\mathbf{x}_*, t) \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_* - t).$$

$$\tilde{\mathbf{A}}(\mathbf{x}_*, t) = (SIR)(t) \mathbf{B}(\mathbf{x}_*).$$

Picture is different...

Original expansion
Is valid only here!

$|y - x_* - t| < r_1 = |t| - r$

Since
 $\Omega_{r_1}(x_*+t) \subset \Omega_r(t)$!

Also
 $|x_i - x_*| < r$
singular point !

Properties of the translation operator

$\mathcal{T}(t)[\Phi(y)] = \Phi(y + t)$

- $\mathcal{T}(0) = \mathcal{I}$ (identity operator). Proof:
 $\mathcal{T}(0)[\Phi(y)] = \Phi(y)$.
- $\mathcal{T}(t_1 + t_2) = \mathcal{T}(t_1) \circ \mathcal{T}(t_2) = \mathcal{T}(t_2) \circ \mathcal{T}(t_1)$. Proof:
 $\mathcal{T}(t_1) \circ \mathcal{T}(t_2)[\Phi(y)] = \Phi(y + t_2 + t_1) = \mathcal{T}(t_2 + t_1)[\Phi(y)] = \mathcal{T}(t_1 + t_2)[\Phi(y)]$.
- (corollary 1): $\mathcal{T}^{-1}(t) = \mathcal{T}(-t)$. Proof:
 $\mathcal{I} = \mathcal{T}(0) = \mathcal{T}(t - t) = \mathcal{T}(t) \circ \mathcal{T}(-t)$.
- (corollary 2): $\mathcal{T}^n(t) = \mathcal{T}(nt)$. Proof (use induction):
 $\mathcal{T}(nt) = \mathcal{T}(n-1)t \circ \mathcal{T}(t) = \mathcal{T}^{n-1}(t) \circ \mathcal{T}(t) = \mathcal{T}^n(t)$.

Spectrum of the translation operator

eigen value \swarrow eigen function

$\mathcal{T}(t)[\Psi(y)] = \lambda \Psi(y), \quad y \in \mathbb{R}^d$

Any function of type

$\forall a \in \mathbb{R}^d, \quad \Psi(y) = e^{ay}, \quad \lambda = e^{at}$

Check:

$\mathcal{T}(t)[\Psi(y)] = \Psi(y + t) = e^{a(y+t)} = e^{at} e^{ay} = \lambda \Psi(y)$

Relation to differential operator:

$\frac{d\Phi(y)}{ds} = \lim_{|t| \rightarrow 0} \frac{\Phi(y+t) - \Phi(y)}{|t|} = \lim_{|t| \rightarrow 0} \frac{\mathcal{T}(t)[\Phi(y)] - \Phi(y)}{|t|} = \lim_{|t| \rightarrow 0} \frac{\mathcal{T}(t) - \mathcal{I}}{|t|}[\Phi(y)], \quad s = \frac{t}{|t|}$

derivative in direction s

Example from previous lectures

$\Phi(y, x_i) = \frac{1}{y - x_i}$

$|y - x_*| < |x_i - x_*|$: R-expansion

$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y - x_*)$

$a_m(x_i, x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, \dots$

$R_m(y - x_*) = (y - x_*)^m, \quad m = 0, 1, \dots$

$|y - x_*| > |x_i - x_*|$: S-expansion

$\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*)$

$b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \dots$

$S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \dots$

In this case we have

$(|y - x_*| < |t|)$

$$S_n(y - x_* + t) = (t + y)^{-n-1} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} (y - x_*)^m$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} R_m(y - x_*) = \sum_{m=0}^{\infty} (S|R)_{mn}(t) R_m(y - x_*).$$

So

$$(S|R)_{mn}(t) = \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} = \frac{(-1)^m (m+n)!}{m! n! t^{m+n+1}}.$$

$$(S|R)(t) = \begin{pmatrix} r^{-1} & r^{-2} & r^{-3} & \dots \\ -r^{-2} & -2r^{-3} & -3r^{-4} & \dots \\ r^{-3} & 3r^{-4} & 6r^{-5} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

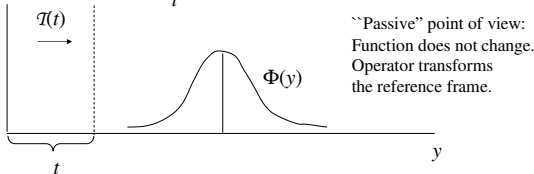
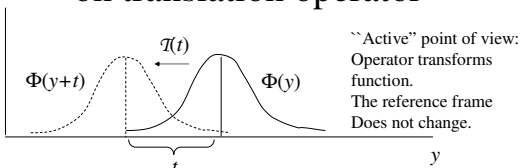
Norm of the Translation Operator

Theorem. Let $\mathbb{F}(\Omega)$ be a set of functions bounded in \mathbb{R}^d . Then $\|\mathcal{T}(t)\| = 1$.

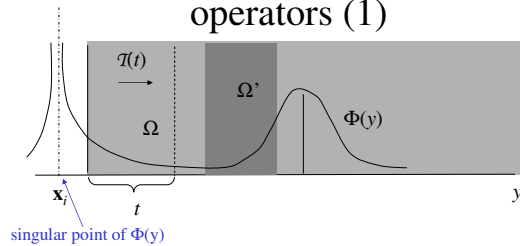
Proof.

$$\|\mathcal{T}(t)\| = \frac{\|\mathcal{T}(t)\Phi(y)\|}{\|\Phi(y)\|} = \frac{\|\Phi(y+t)\|}{\|\Phi(y)\|} = \frac{\sup_{y \in \mathbb{R}^d} |\Phi(y+t)|}{\sup_{y \in \mathbb{R}^d} |\Phi(y)|} = 1.$$

Active and Passive points of view on translation operator



Norms of R|R, S|S, and S|R-operators (1)



$\Phi(y)$ is bounded in Ω .
 $\Omega' \subset \Omega$.
Therefore $\Phi(y)$ is bounded in Ω' , and

$$\|\Phi(y)\|_{\Omega'} = \sup_{y \in \Omega'} |\Phi(y)| \leq \sup_{y \in \Omega} |\Phi(y)| = \|\Phi(y)\|_{\Omega}$$

Norms of $R|R$, $S|S$, and $S|R$ -operators (2)

From the passive point of view, the translation operator does nothing, but just changes the reference frame. So if we consider that $R|R$, $S|S$, and $S|R$ do just change of the reference frame **PLUS they shrink the domain, where the function is bounded, then their norms do not exceed 1.**

$$\Omega' \subset \Omega$$

$$\|(\mathcal{R}|\mathcal{R})(t)\| = \frac{\sup_{y \in \Omega'} |\Phi(y)|}{\sup_{y \in \Omega} |\Phi(y)|} \leq 1,$$

$$\|(\mathcal{S}|\mathcal{S})(t)\| = \frac{\sup_{y \in \Omega'} |\Phi(y)|}{\sup_{y \in \Omega} |\Phi(y)|} \leq 1,$$

$$\|(\mathcal{S}|\mathcal{R})(t)\| = \frac{\sup_{y \in \Omega'} |\Phi(y)|}{\sup_{y \in \Omega} |\Phi(y)|} \leq 1.$$

This is the difference between general translation operator and $R|R$, $S|S$, and $S|R$ operators.

Error of exact $R|R$, $S|S$, and $S|R$ -translation

If

$$\|\Phi(y) - \Phi^p(y)\| < \epsilon,$$

then

$$\|(\mathcal{R}|\mathcal{R})(t)(\Phi(y) - \Phi^p(y))\| = \|(\mathcal{R}|\mathcal{R})(t)\| \|\Phi(y) - \Phi^p(y)\| < \epsilon,$$

$$\|(\mathcal{S}|\mathcal{S})(t)(\Phi(y) - \Phi^p(y))\| = \|(\mathcal{S}|\mathcal{S})(t)\| \|\Phi(y) - \Phi^p(y)\| < \epsilon,$$

$$\|(\mathcal{S}|\mathcal{R})(t)(\Phi(y) - \Phi^p(y))\| = \|(\mathcal{S}|\mathcal{R})(t)\| \|\Phi(y) - \Phi^p(y)\| < \epsilon.$$