FMM CMSC 878R/AMSC 698R

Lecture 5

Outline

- Factorization of Scalar Products in $\mathbf{R}^d$ (compression)
  - Factorization in 2D.
  - Factorization in 3D.
  - Factorization in *d*D.
  - Multinomial Coefficients.
  - Length of compressed vector.
  - Example.
  - Complexity of Fast Summation.

Compression

Compression operator:

$$\mathbf{A}^n = \mathbf{Compress}(\mathbf{a}^n)$$

Required Property:

$$\mathbf{a}^n \bullet \mathbf{b}^n = \mathbf{Compress}(\mathbf{a}^n) \bullet \mathbf{Compress}(\mathbf{b}^n).$$

Consider $\mathbf{R}^2$:

$$\mathbf{a}^n \bullet \mathbf{b}^n = (\mathbf{a} \bullet \mathbf{b})^n = (a_1 b_1 + a_2 b_2)^n$$

$$= a_1^n b_1^n + \binom{n}{1} a_1^{n-1} b_1^{n-1} a_2 b_2 + \binom{n}{2} a_1^{n-2} b_1^{n-2} a_2^2 b_2^2 + \ldots + a_2^n b_2^n$$

The length is only $(n+1)$, not 2^n

Let us define:

$$\mathbf{A}^n = \mathbf{Compress}(\mathbf{a}^n) = \left(a_1^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_2, \sqrt{\binom{n}{2}} a_1^{n-2} a_2^2, \ldots, a_2^n \right),$$

$$\mathbf{B}^n = \mathbf{Compress}(\mathbf{b}^n) = \left(b_1^n, \sqrt{\binom{n}{1}} b_1^{n-1} b_2, \sqrt{\binom{n}{2}} b_1^{n-2} b_2^2, \ldots, b_2^n \right)$$

Compression Can be Performed for any Dimensionality (Example for 3D):

$$\mathbf{a}^n \bullet \mathbf{b}^n = (\mathbf{a} \bullet \mathbf{b})^n = (a_1 b_1 + a_2 b_2 + a_3 b_3)^n$$

$$= [(a_1 b_1 + a_2 b_2) + a_3 b_3]^n = \sum_{m=0}^{n} \binom{n}{m} (a_1 b_1 + a_2 b_2)^{n-m} a_3^m b_3^m$$

$$= \sum_{m=0}^{n} \sum_{l=0}^{n-m} \binom{n}{m} \binom{n-m}{l} a_1^{n-m-l} b_1^{n-m-l} a_2^l b_2^l a_3^m b_3^m$$

$$= a_1^n b_1^n + \binom{n}{1} a_1^{n-1} b_1^{n-1} a_2 b_2 + \binom{n}{2} a_1^{n-2} b_1^{n-2} a_2^2 b_2^2 + \ldots + a_2^n b_2^n$$

$$+ \binom{n}{1} a_1^{n-1} b_1^{n-1} a_3 b_3 + \binom{n}{1} \binom{n-1}{1} a_1^{n-2} b_1^{n-2} a_2 b_2 a_3 b_3 + \ldots + a_3^n b_3^n,$$

$$\mathbf{Compress}(\mathbf{a}^n) = \left(a_1^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_2, \sqrt{\binom{n}{2}} a_1^{n-2} a_2^2, \ldots, a_2^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_3, \ldots, a_3^n \right)$$

The length of $\mathbf{a}^n$ is $(n+1)+n+\ldots+1=(n+1)(n+2)/2$

Compression Can be Performed for any Dimensionality (General Case):

$$(a_1 + a_2 + \ldots + a_d)^n = \sum_{n_1+\ldots+n_d=n} (n; n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d}.$$

$$(n; n_1, n_2, \ldots, n_d) = \frac{n!}{n_1! n_2! \ldots n_d!}.$$

Multinomial coefficients

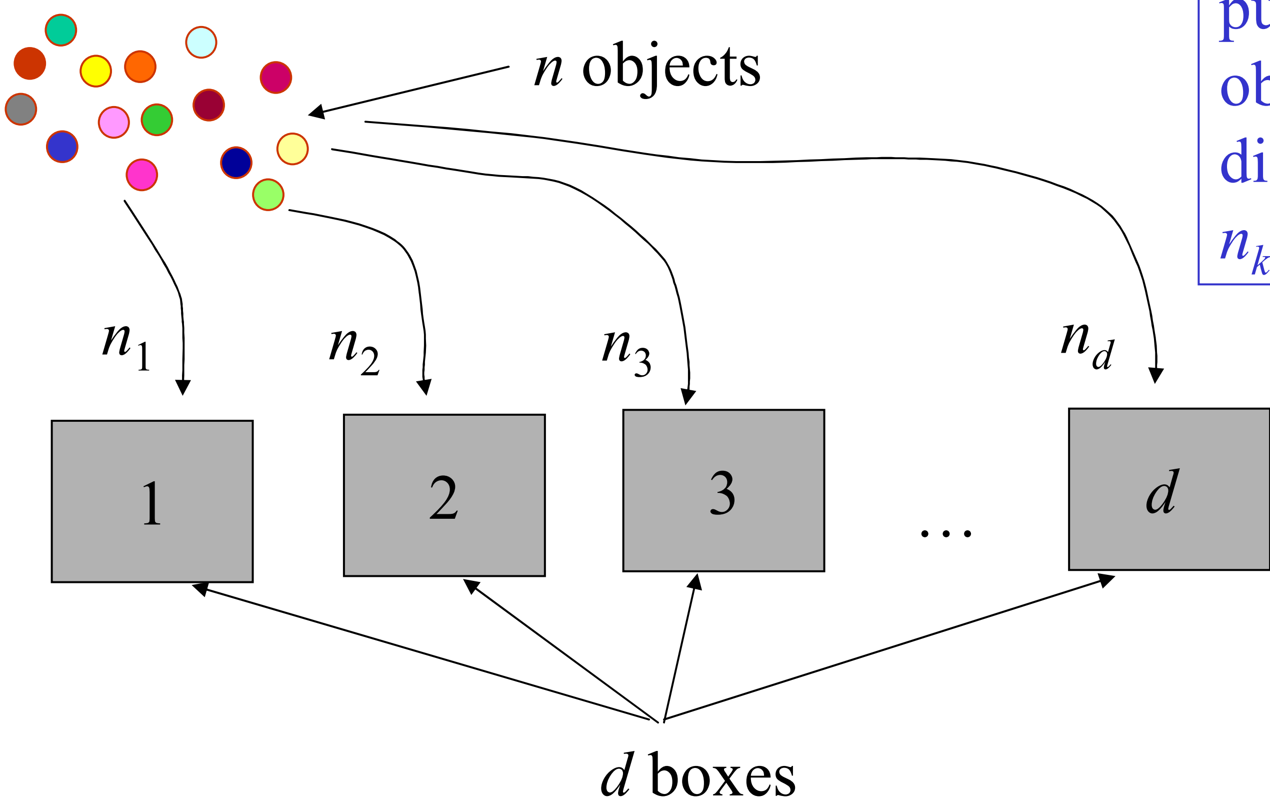
$$\mathbf{Compress}(\mathbf{a}^n) = \left(a_1^n, \sqrt{(n; n-1, 1, 0, \ldots, 0)}\, a_1^{n-1} a_2, \ldots, \sqrt{(n; n_1, n_2, \ldots, n_d)}\, a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d}, \ldots, a_d^n \right)$$

So we have

$$\begin{aligned}\mathbf{a}^n \cdot \mathbf{b}^n &= \mathbf{Compress}(\mathbf{a}^n) \cdot \mathbf{Compress}(\mathbf{b}^n) \\ &= \sum_{n_1+\ldots+n_d=n} (n; n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d} b_1^{n_1} b_2^{n_2} \ldots b_d^{n_d} \\ &= (a_1 b_1 + a_2 b_2 + \ldots + a_d b_d)^n = (\mathbf{a} \cdot \mathbf{b})^n.\end{aligned}$$

What are multinomial coefficients?

$(n\,;\,n_1,n_2,\ldots,n_d)$ is the number of ways of putting n different objects into d different boxes with n_k in the k-th box



$$n_1 + n_2 + \ldots + n_d = n$$

The length of the compressed vector

$$
\begin{aligned}
d = 1: &\quad 1, \\
d = 2: &\quad n+1, \\
d = 3: &\quad \frac{1}{2}(n+1)(n+2), \\
&\dots
\end{aligned}
$$

Theorem: If $\mathbf{a} \in \mathbb{R}^d$, then the length of compressed vector $\mathbf{Compress}(\mathbf{a}^n)$, is

$$\binom{n+d-1}{n} = \frac{(n+1)\dots(n+d-1)}{(d-1)!}.$$

Proof: We have a basis for induction (see above). Let this holds for d dimensions. Consider $d+1$ dimensions:

$$((a_1 + \dots + a_d) + a_{d+1})^n = \sum_{m=0}^{n} \binom{n}{m} (a_1 + \dots + a_d)^m a_{d+1}^{n-m}$$

The number of terms is then

$$\sum_{m=0}^{n} \binom{m+d-1}{m} = \binom{d-1}{0} + \binom{d}{1} + \dots + \binom{n+d-1}{n} = \binom{n+d}{n}$$

This proves the theorem.

Example of Fast Computation

$$v_j = \sum_{i=1}^{N} u_i \Phi(\mathbf{y}_j, \mathbf{x}_i) = \sum_{m=0}^{p-1} \mathbf{c}_m \bullet (\mathbf{y}_j - \mathbf{x}_*)^m + \text{Residual}, \qquad \mathbf{c}_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} \mathbf{x}_i^m.$$

Equivalent to:

$$v_j = \sum_{m=0}^{p-1} \mathbf{C}_m \bullet \mathbf{Compress}\left((\mathbf{y}_j - \mathbf{x}_*)^m \right) + \text{Residual}, \qquad \mathbf{C}_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} \mathbf{Compress}(\mathbf{x}_i^m).$$

Number of multiplications (complexity) to obtain v_j: (in 2D case!)

$$Complexity = 1 + 2 + \ldots + p = \frac{p(p+1)}{2}.$$

$$\mathbf{C}_0 = \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i},$$

$$\mathbf{C}_1 = (C_{11}, C_{12}) = \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} (x_{i1}, x_{i2}),$$

$$\mathbf{C}_2 = (C_{21}, C_{22}, C_{23}) = \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} \left(x_{i1}^2, \sqrt{2} x_{i1} x_{i2}, x_{i2}^2 \right),$$

Complexity of Fast Summation

Let $\circ$ be a scalar product of vectors $\mathbf{A}_i$ and $\mathbf{F}_j$ of length $P(p)$ (p is the truncation number).
Complexity of summation over i is then $O(PN)$.
Complexity of scalar product operation is P.
Complexity of M scalar product operations is $O(PM)$ (for $j = 1, ..., M$).
Total complexity is $O(PM + PN)$.
Fast Method is more efficient than direct only if $O(PM + PN) < O(MN)$,
so we should have

$$P(p) \ll \min(M, N)$$