## Is That a Factorization?

$$e^{\mathbf{y}\cdot\mathbf{x}_i} = e^{\mathbf{x}_*\cdot\mathbf{x}_i}\sum_{m=0}^{\infty}\frac{1}{m!}[(\mathbf{y}-\mathbf{x}_*)\cdot\mathbf{x}_i]^m$$

## Scalar Product in d-Dimensional Space

Definition of scalar product:

$$\mathbf{a} = (a_1,\ldots,a_d), \quad \mathbf{b} = (b_1,\ldots,b_d),$$

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + \ldots + a_d b_d = \sum_{k=1}^{d} a_k b_k.$$

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta,$$

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}.$$

What if

$$a_1,\ldots,a_d,b_1,\ldots,b_d \in \mathbb{C} \quad ?$$

Definition:

complex conjugate

$$\mathbf{a}\cdot\mathbf{b} = \overline{a_1} b_1 + \ldots + \overline{a_d} b_d = \sum_{k=1}^{d} \overline{a_k} b_k.$$

## Properties of Scalar Product

Commutativity:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

Scaling:

$$(\lambda\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(\lambda\mathbf{b}) = \lambda(\mathbf{a}\cdot\mathbf{b}), \quad \lambda \in \mathrm{R}$$

Distributivity:

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$$

## Factorization of Scalar Product Powers

$$(\mathbf{a}\cdot\mathbf{b})^n = \left(\sum_{k=1}^{d} a_k b_k\right)^n = \sum_{k_1=1}^{d} a_{k_1} b_{k_1} \sum_{k_2=1}^{d} a_{k_2} b_{k_2} \ldots \sum_{k_n=1}^{d} a_{k_n} b_{k_n}$$

$$= \sum_{k_1=1}^{d}\sum_{k_2=1}^{d}\ldots\sum_{k_n=1}^{d} a_{k_1} a_{k_2}\ldots a_{k_n} b_{k_1} b_{k_2}\ldots b_{k_n}$$

$$= [\mathbf{a}\otimes\mathbf{a}\otimes\ldots\otimes\mathbf{a}]\cdot[\mathbf{b}\otimes\mathbf{b}\otimes\ldots\otimes\mathbf{b}] = \mathbf{a}^n\cdot\mathbf{b}^n$$

$$\mathbf{a}^n\cdot\mathbf{b}^n = (\mathbf{a}\cdot\mathbf{b})^n = (\mathbf{b}\cdot\mathbf{a})^n = \mathbf{b}^n\cdot\mathbf{a}^n.$$

$$e^{\mathbf{y}\cdot\mathbf{x}_i} = e^{\mathbf{x}_*\cdot\mathbf{x}_i}\sum_{m=0}^{\infty}\frac{1}{m!}[(\mathbf{y}-\mathbf{x}_*)\cdot\mathbf{x}_i]^m = e^{\mathbf{x}_*\cdot\mathbf{x}_i}\sum_{m=0}^{\infty}\frac{1}{m!}\mathbf{x}_i^m\cdot(\mathbf{y}-\mathbf{x}_*)^m.$$

## Is That Factorization?

1) Truncation:

$$\Phi(\mathbf{y},\mathbf{x}_i) = e^{\mathbf{y}\bullet\mathbf{x}_i} = e^{\mathbf{x}_*\bullet\mathbf{x}_i}\left[\sum_{m=0}^{p-1}\frac{1}{m!}\mathbf{x}_i^m \bullet (\mathbf{y}-\mathbf{x}_*)^m + \text{Residual}_p\right]$$

2) Fast summation:

$$v_j = \sum_{i=1}^{N} u_i \Phi(\mathbf{y}_j,\mathbf{x}_i) = \sum_{i=1}^{N} u_i e^{\mathbf{x}_*\bullet\mathbf{x}_i}\left[\sum_{m=0}^{p-1}\frac{1}{m!}\mathbf{x}_i^m \bullet (\mathbf{y}_j-\mathbf{x}_*)^m + \text{Residual}_p\right]$$

$$= \sum_{i=1}^{N} u_i e^{\mathbf{x}_*\bullet\mathbf{x}_i}\sum_{m=0}^{p-1}\frac{1}{m!}\mathbf{x}_i^m \bullet (\mathbf{y}_j-\mathbf{x}_*)^m + N\max_i(u_i e^{\mathbf{x}_*\bullet\mathbf{x}_i})\text{Residual}_p$$

$$= \sum_{m=0}^{p-1}\frac{1}{m!}\left(\sum_{i=1}^{N} u_i e^{\mathbf{x}_*\bullet\mathbf{x}_i}\mathbf{x}_i^m\right) \bullet (\mathbf{y}_j-\mathbf{x}_*)^m + \text{Residual}$$

$$= \sum_{m=0}^{p-1}\mathbf{c}_m \bullet (\mathbf{y}_j-\mathbf{x}_*)^m + \text{Residual}, \qquad \mathbf{c}_m = \frac{1}{m!}\sum_{i=1}^{N} u_i e^{\mathbf{x}_*\bullet\mathbf{x}_i}\mathbf{x}_i^m.$$

Yes! It is!

## Example (Let's Try To Get Explicit Forms in 2D)

$$\mathbf{a} = (a_1,a_2),$$

$$\mathbf{a}^2 = (a_1(a_1,a_2),a_2(a_1,a_2)) = (a_1^2,a_1a_2,a_2a_1,a_2^2),$$

$$\mathbf{a}^3 = (a_1^2(a_1,a_2),a_1a_2(a_1,a_2),a_2a_1(a_1,a_2),a_2^2(a_1,a_2))$$

$$= (a_1^3,a_1^2a_2,a_1a_2a_1,a_1a_2^2,a_2a_1^2,a_2a_1a_2,a_2^2a_1,a_2^3),\ldots$$

The length of $\mathbf{a}^n$ is $2^n$! ← This is not factorial!

In $d$ dimensions the length of $\mathbf{a}^n$ is even $d^n$

What to do in practical problems?

## Use Compression!

Compression operator:

$$\mathbf{A}^n = \mathbf{Compress}(\mathbf{a}^n)$$

Required Property:

$$\mathbf{a}^n \bullet \mathbf{b}^n = \mathbf{Compress}(\mathbf{a}^n) \bullet \mathbf{Compress}(\mathbf{b}^n).$$

Consider $\mathbf{R}^2$:

$$\mathbf{a}^n \bullet \mathbf{b}^n = (\mathbf{a}\bullet\mathbf{b})^n = (a_1b_1+a_2b_2)^n$$

$$= a_1^nb_1^n + \binom{n}{1}a_1^{n-1}b_1^{n-1}a_2b_2 + \binom{n}{2}a_1^{n-2}b_1^{n-2}a_2^2b_2^2 + \ldots + a_2^nb_2^n$$

The length is only $(n+1)$, not $2^n$

Let us define:

$$\mathbf{A}^n = \mathbf{Compress}(\mathbf{a}^n) = \left(a_1^n, \sqrt{\binom{n}{1}}a_1^{n-1}a_2, \sqrt{\binom{n}{2}}a_1^{n-2}a_2^2,\ldots,a_2^n\right),$$

$$\mathbf{B}^n = \mathbf{Compress}(\mathbf{b}^n) = \left(b_1^n, \sqrt{\binom{n}{1}}b_1^{n-1}b_2, \sqrt{\binom{n}{2}}b_1^{n-2}b_2^2,\ldots,b_2^n\right)$$

## Example of Fast Computation

$$v_j = \sum_{i=1}^{N} u_i \Phi(\mathbf{y}_j,\mathbf{x}_i) = \sum_{m=0}^{p-1}\mathbf{c}_m \bullet (\mathbf{y}_j-\mathbf{x}_*)^m + \text{Residual}, \qquad \mathbf{c}_m = \frac{1}{m!}\sum_{i=1}^{N} u_i e^{\mathbf{x}_*\bullet\mathbf{x}_i}\mathbf{x}_i^m.$$

Equivalent to:

$$v_j = \sum_{m=0}^{p-1}\mathbf{C}_m \bullet \mathbf{Compress}\left((\mathbf{y}_j-\mathbf{x}_*)^m\right) + \text{Residual}, \qquad \mathbf{C}_m = \frac{1}{m!}\sum_{i=1}^{N} u_i e^{\mathbf{x}_*\bullet\mathbf{x}_i}\mathbf{Compress}(\mathbf{x}_i^m).$$

Number of multiplications (complexity) to obtain $v_j$:

$$\mathit{Complexity} = 1+2+\ldots+p = \frac{p(p+1)}{2}.$$

## Compression Can be Performed for any Dimensionality (Example for 3D):

$$\mathbf{a}^n \bullet \mathbf{b}^n = (\mathbf{a} \bullet \mathbf{b})^n = (a_1 b_1 + a_2 b_2 + a_3 b_3)^n$$

$$= [(a_1 b_1 + a_2 b_2) + a_3 b_3]^n = \sum_{m=0}^{n} \binom{n}{m} (a_1 b_1 + a_2 b_2)^{n-m} a_3^m b_3^m$$

$$= \sum_{m=0}^{n} \sum_{l=0}^{n-m} \binom{n}{m} \binom{n-m}{l} a_1^{n-m-l} b_1^{n-m-l} a_2^l b_2^l a_3^m b_3^m$$

$$= a_1^n b_1^n + \binom{n}{1} a_1^{n-1} b_1^{n-1} a_2 b_2 + \binom{n}{2} a_1^{n-2} b_1^{n-2} a_2^2 b_2^2 + \ldots + a_2^n b_2^n$$

$$+ \binom{n}{1} a_1^{n-1} b_1^{n-1} a_3 b_3 + \binom{n}{1} \binom{n-1}{1} a_1^{n-2} b_1^{n-2} a_2 b_2 a_3 b_3 + \ldots + a_3^n b_3^n,$$

$$\mathbf{Compress}(\mathbf{a}^n) = \left( a_1^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_2, \sqrt{\binom{n}{2}} a_1^{n-2} a_2^2, \ldots, a_2^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_3, \ldots, a_3^n \right)$$

The length of $\mathbf{a}^n$ is $(n+1)+n+\ldots+1 = (n+1)(n+2)/2$

## Compression Can be Performed for any Dimensionality (General Case):

$$(a_1 + a_2 + \ldots + a_d)^n = \sum_{n_1 + \ldots + n_d = n} (n; n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d}.$$

$$(n; n_1, n_2, \ldots, n_d) = \frac{n!}{n_1! n_2! \ldots n_d!}.$$

Multinomial coefficients

$$\mathbf{Compress}(\mathbf{a}^n) = \left( a_1^n, \sqrt{(n; n-1, 1, 0, \ldots, 0)} a_1^{n-1} a_2, \ldots, \sqrt{(n; n_1, n_2, \ldots, n_d)} a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d}, \ldots, a_d^n \right)$$

So we have

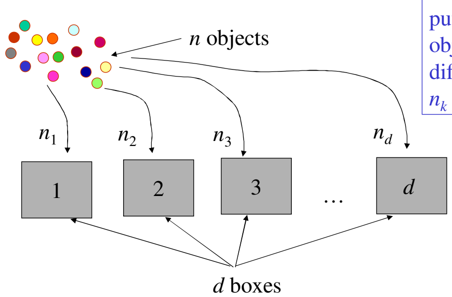
$$\mathbf{a}^n \cdot \mathbf{b}^n = \mathbf{Compress}(\mathbf{a}^n) \cdot \mathbf{Compress}(\mathbf{b}^n)$$

$$= \sum_{n_1 + \ldots + n_d = n} (n; n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d} b_1^{n_1} b_2^{n_2} \ldots b_d^{n_d}$$

$$= (a_1 b_1 + a_2 b_2 + \ldots + a_d b_d)^n = (\mathbf{a} \cdot \mathbf{b})^n.$$

## What are multinomial coefficients?

$(n; n_1, n_2, \ldots, n_d)$ is the number of ways of putting $n$ different objects into $d$ different boxes with $n_k$ in the $k$-th box




$n_1 + n_2 + \ldots + n_d = n$

## The length of the compressed vector

$$d = 1: \quad 1,$$
$$d = 2: \quad n+1,$$
$$d = 3: \quad \tfrac{1}{2}(n+1)(n+2),$$
$$\ldots$$

**Theorem**: If $\mathbf{a} \in \mathbb{R}^d$, then the length of compressed vector $\mathbf{Compress}(\mathbf{a}^n)$, is

$$\binom{n+d-1}{n} = \frac{(n+1)\ldots(n+d-1)}{(d-1)!}.$$

**Proof**: We have a basis for induction (see above). Let this holds for $d$ dimensions. Consider $d+1$ dimensions:

$$((a_1 + \ldots + a_d) + a_{d+1})^n = \sum_{m=0}^{n} \binom{n}{m} (a_1 + \ldots + a_d)^m a_{d+1}^{n-m}$$

The number of terms is then

$$\sum_{m=0}^{n} \binom{m+d-1}{m} = \binom{d-1}{0} + \binom{d}{1} + \ldots + \binom{n+d-1}{n} = \binom{n+d}{n}$$

This proves the theorem.

# Example of Fast Computation

$$v_j = \sum_{i=1}^{N} u_i \Phi(\mathbf{y}_j, \mathbf{x}_i) = \sum_{m=0}^{p-1} \mathbf{c}_m \bullet (\mathbf{y}_j - \mathbf{x}_*)^m + \text{Residual}, \qquad \mathbf{c}_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} \mathbf{x}_i^m.$$

Equivalent to:

$$v_j = \sum_{m=0}^{p-1} \mathbf{C}_m \bullet \mathbf{Compress}\left((\mathbf{y}_j - \mathbf{x}_*)^m\right) + \text{Residual}, \qquad \mathbf{C}_m = \frac{1}{m!} \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} \mathbf{Compress}(\mathbf{x}_i^m).$$

Number of multiplications (complexity) to obtain $v_j$: (in 2D case!)

$$Complexity = 1 + 2 + \ldots + p = \frac{p(p+1)}{2}.$$

$$\mathbf{C}_0 = \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i},$$

$$\mathbf{C}_1 = (C_{11}, C_{12}) = \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} (x_{i1}, x_{i2}),$$

$$\mathbf{C}_2 = (C_{21}, C_{22}, C_{23}) = \sum_{i=1}^{N} u_i e^{\mathbf{x}_* \bullet \mathbf{x}_i} \left(x_{i1}^2, \sqrt{2} x_{i1} x_{i2}, x_{i2}^2\right),$$

# Complexity of Fast Summation

Let $\circ$ be a scalar product of vectors $A_i$ and $F_j$ of length $P(p)$ ($p$ is the truncation number).
Complexity of summation over $i$ is then $O(PN)$.
Complexity of scalar product operation is $P$.
Complexity of $M$ scalar product operations is $O(PM)$ (for $j = 1, \ldots, M$).
Total complexity is $O(PM + PN)$.
Fast Method is more efficient than direct only if $O(PM + PN) < O(MN)$,
so we should have

$$P(p) \ll \min(M, N)$$

# General Forms of Factorization for Fast Summation (1)

$$v_j = \sum_{i=1}^{N} u_i \Phi(\mathbf{y}_j, \mathbf{x}_i), \quad j = 1, \ldots, M.$$

$$\Phi(\mathbf{y}_j, \mathbf{x}_i) = \sum_{m=0}^{p} a_m(\mathbf{x}_i, \mathbf{x}_*) f_m(\mathbf{y}_j - \mathbf{x}_*) + Error(p, \mathbf{x}_i, \mathbf{x}_*, \mathbf{y}_j)$$
$$= \mathbf{a}(\mathbf{x}_i, \mathbf{x}_*) \bullet \mathbf{f}(\mathbf{y}_j - \mathbf{x}_*) + Error.$$

scalar product

vectors of length $p$

How about

$$v_j = \sum_{i=1}^{N} u_i e^{-\lambda_i |\mathbf{x}_i - \mathbf{y}_j|^2}$$

Some parameter depending on $i$

More general to have

$$v_j = \sum_{i=1}^{N} u_i \Phi_i(\mathbf{y}_j) \quad \text{or} \quad v(\mathbf{y}) = \sum_{i=1}^{N} u_i \Phi_i(\mathbf{y}).$$

# General Forms of Factorization for Fast Summation (2)

The potential can be factorized as

$$\Phi_i(\mathbf{y}) = \mathbf{A}_i(\mathbf{x}_*) \circ \mathbf{F}(\mathbf{y} - \mathbf{x}_*)$$

Generalized product $\circ$ can be scalar product, contraction, etc. $\mathbf{A}_i$ and $\mathbf{F}$ can be real or complex vectors, tensors, etc. in $p$-dimensional space.
Requirements to the product (distributivity with respect to addition)

$$(\alpha \mathbf{A}_i + \beta \mathbf{A}_j) \circ \mathbf{F} = \alpha \mathbf{A}_i \circ \mathbf{F} + \beta \mathbf{A}_j \circ \mathbf{F}.$$

In this case

$$v(\mathbf{y}) = \sum_{i=1}^{N} u_i \Phi_i(\mathbf{y}) = \sum_{i=1}^{N} u_i \mathbf{A}_i(\mathbf{x}_*) \circ \mathbf{F}(\mathbf{y} - \mathbf{x}_*) = \mathbf{A}(\mathbf{x}_*) \circ \mathbf{F}(\mathbf{y} - \mathbf{x}_*)$$

$$\mathbf{A}(\mathbf{x}_*) = \sum_{i=1}^{N} u_i \mathbf{A}_i(\mathbf{x}_*)$$

We do not need commutativity of $\circ$ (i.e. we do not request $\mathbf{A}_i \circ \mathbf{F} = \mathbf{F} \circ \mathbf{A}_i$)(!).

# General Forms of Factorization for Fast Summation (3)

Actually, we even do need continuous variable **y**,
The problem is to represent all matrix elements in the form

$$\Phi_{ji} = \mathbf{A}_i \circ \mathbf{F}_j$$

then

$$v_j = \sum_{i=1}^{N} u_i \Phi_{ji} = \sum_{i=1}^{N} u_i (\mathbf{A}_i \circ \mathbf{F}_j) = \left( \sum_{i=1}^{N} u_i \mathbf{A}_i \right) \circ \mathbf{F}_j.$$

# Outline

- Far Field Expansions (or S-expansions)
  - Regular Potential (Convergent Series);
  - Regular Potential (Asymptotic Series);
  - Singular Potential;
- Asymptotic Series
- Approaches for Selection of the Basis Functions

# Far Field Expansions (S-expansions)

Let

$$\mathbf{x}_* \in \mathrm{R}^d.$$

Might be Singular (at $\mathbf{y} = \mathbf{x}_*$) Basis Functions

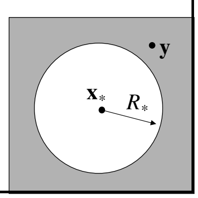
We call expansion

$$\Phi(\mathbf{y}, \mathbf{x}_i) = \sum_{m=0}^{\infty} b_m(\mathbf{x}_i, \mathbf{x}_*) S_m(\mathbf{y} - \mathbf{x}_*)$$

far field expansion (or S-expansion) outside a sphere

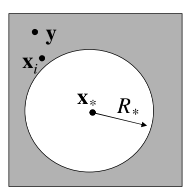
$$|\mathbf{y} - \mathbf{x}_*| > R_*,$$

if the series converges for $\forall \mathbf{y},\ |\mathbf{y} - \mathbf{x}_*| > R_*$.



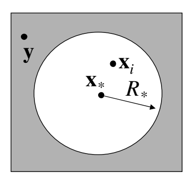

# Far Field Expansion of a Regular Potential

…sometimes like this:



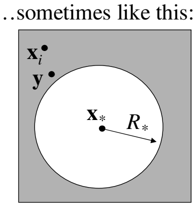

$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$

Can be like this:




$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$

…sometimes like this:




$|\mathbf{x}_i - \mathbf{x}_*| > |\mathbf{y} - \mathbf{x}_*| > R_*$

## Local Expansion of a Regular Potential Can be Far Field Expansion Also

Valid for any $r_* < \infty$, and $x_i$.

$$\Phi(y,x_i) = e^{-(y-x_i)^2} = \sum_{m=0}^{\infty} a_m(x_i,x_*)S_m(y-x_*).$$

We have

$$e^{-(y-x_i)^2} = \sum_{m=0}^{\infty} S_m(x_i-x_*)R_m(y-x_*) = \sum_{m=0}^{\infty} R_m(x_i-x_*)S_m(y-x_*)$$

$$R_m(x) = S_m(x) = \sqrt{\frac{2^m}{m!}}\, e^{-x^2}x^m, \quad m = 0,1,\ldots$$

## Or may be not…

Decays at large y

$$e^{-(y-x_i)^2} = \sum_{l=0}^{\infty} h_l(x_i-x_*)\frac{(y-x_*)^l}{l!} = \sum_{l=0}^{\infty} \frac{(x_i-x_*)^l}{l!}h_l(y-x_*)$$

Local

Far

## Asymptotic Series

$$f(x,\epsilon) = f_0(x)\varphi_0(\epsilon) + f_1(x)\varphi_1(\epsilon) + f_2(x)\varphi_2(\epsilon) + \ldots = \sum_{n=0}^{\infty} f_n(x)\varphi_n(\epsilon)$$

$$\lim_{\epsilon\to 0}\frac{\varphi_n(\epsilon)}{\varphi_{n+1}(\epsilon)} = 0.$$

Gauge functions

$$f(x,\epsilon) - \sum_{n=0}^{p-1} f_n(x)\varphi_n(\epsilon) = O(f_p(x)\varphi_p(\epsilon))$$

The asymptotic expansion is *uniform* in domain $x \in \Omega$ if

$$\forall x \in \Omega, \quad \left| f(x,\epsilon) - \sum_{n=0}^{p-1} f_n(x)\varphi_n(\epsilon)\right| = O(\varphi_p(\epsilon)).$$

Otherwise the asymptotic expansion is not uniform.

## Examples of Uniform and Non-Uniform Expansions

Example of uniform expansion:

$$f(x,\epsilon) = \frac{1}{x+\epsilon}, \quad x > 10$$

$$f(x,\epsilon) = \frac{1}{x}\left(1+\frac{\epsilon}{x}\right)^{-1} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n\epsilon^n}{x^n}$$

Example of non-uniform expansion:

$$f(x,\epsilon) = e^{\epsilon x}, \quad x \in \mathbb{R}^1$$

$$e^{\epsilon x} = \sum_{n=0}^{\infty}\frac{\epsilon^n x^n}{n!}.$$

Prove that! (Hint: consider $x \gg \epsilon^{-1}$).

## Example of Far Field Expansion of a Regular Function (Using Asymptotic Series)

$$\Phi(y,x_i) = \frac{1}{1+(y-x_i)^2} = \frac{1}{1+[y-x_*-(x_i-x_*)]^2} = \frac{1}{(y-x_*)^2}\frac{(y-x_*)^2}{1+[y-x_*-(x_i-x_*)]^2}.$$

Let

$$\epsilon = \frac{1}{y-x_*}$$

$$\Phi(\epsilon,x_i-x_*) = \epsilon^2\frac{\frac{1}{\epsilon^2}}{1+\left[\frac{1}{\epsilon}-(x_i-x_*)\right]^2} = \epsilon^2\frac{1}{\epsilon^2+(1-\epsilon x)^2} = \epsilon^2 f(x,\epsilon), \quad x = x_i - x_*$$

$$f(x,\epsilon) = \frac{1}{\epsilon^2+(1-\epsilon x)^2} = \sum_{n=0}^{\infty} f_n(x)\epsilon^n$$

$$f_n(x) = \frac{1}{n!}\frac{\partial^n f(x,\epsilon)}{\partial \epsilon^n}\bigg|_{\epsilon=0}$$

## Example of Far Field Expansion of a Regular Function (continuation)

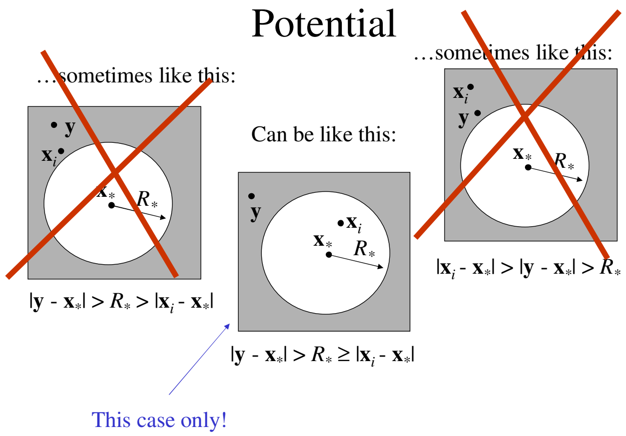
$$f_0(x) = 1,$$
$$f_1(x) = \frac{\partial f(x,\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} = 2x,$$
$$f_2(x) = \frac{1}{2!}\frac{\partial^2 f(x,\epsilon)}{\partial \epsilon^2}\bigg|_{\epsilon=0} = 3x^2-1,$$

$$\Phi(y,x_i) = \frac{1}{(y-x_*)^2}\sum_{n=0}^{\infty} f_n(x_i-x_*)\frac{1}{(y-x_*)^n}$$

$$y \geqslant 100, \quad x_i = 1, \quad x_* = 0,$$
$$\epsilon = 10^{-2}, \quad x = 1$$

$$\left|\Phi(y,x_i) - \frac{1}{(y-x_*)^2}\left[1+\frac{2(x_i-x_*)}{(y-x_*)}\right]\right| \leqslant \epsilon^4(3x^2-1) = 2\cdot 10^{-8}.$$

## Far Field Expansion of a Singular Potential




## Example For S-expansion of Singular Potential

$$\Phi(y,x_i) = \frac{1}{y-x_i}.$$

$$\frac{1}{y-x_i} = \frac{1}{y-x_*-(x_i-x_*)} = \frac{1}{(y-x_*)\left[1-\frac{x_i-x_*}{y-x_*}\right]} = \frac{1}{(y-x_*)}\left[1-\frac{x_i-x_*}{y-x_*}\right]^{-1}.$$

$$\left[1-\frac{x_i-x_*}{y-x_*}\right]^{-1} = \sum_{m=0}^{\infty}\frac{(x_i-x_*)^m}{(y-x_*)^m}, \quad |y-x_*| > |x_i-x_*|.$$

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} b_m(x_i,x_*)S_m(y-x_*),$$
$$b_m(x_i,x_*) = (x_i-x_*)^m, \quad m = 0,1,\ldots,$$
$$S_m(y-x_*) = (y-x_*)^{-m-1}, \quad m = 0,1,\ldots$$

# Let us compare with the R-expansion of the same function

$|y - x_*| < |x_i - x_*|$ :

R-expansion

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} a_m(x_i,x_*) R_m(y-x_*),$$

$$a_m(x_i,x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, \ldots,$$

$$R_m(y-x_*) = (y - x_*)^m, \quad m = 0, 1, \ldots$$

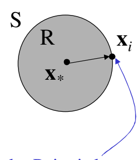
$|y - x_*| > |x_i - x_*|$ :

S-expansion

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} b_m(x_i,x_*) S_m(y-x_*),$$

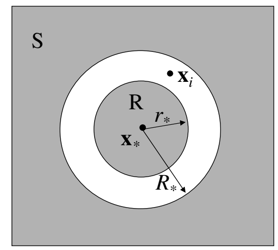
$$b_m(x_i,x_*) = (x_i - x_*)^m, \quad m = 0, 1, \ldots,$$

$$S_m(y-x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \ldots$$




Singular Point is located at the Boundary of regions for the R- and S-expansions!

# What Do We Need For Real FMM (that provides spatial grouping)




$r_* < R_*$

We need S-expansion for $|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$

We need R-expansion for $|\mathbf{y} - \mathbf{x}_*| < r_* < |\mathbf{x}_i - \mathbf{x}_*|$

# Basis Functions

- Power series are great, but do they provide the best approximation? (sometimes yes!)
- Other approaches to factorization:
  - Asymptotic Series (Can be divergent!);
  - Orthogonal Bases in $L_2$;
  - Eigen Functions of Differential Operators;
  - Functions Generated by Differentiation or Other Linear Operators.
- Some of this approaches will be considered in this course.