Is That a Factorization?

\[ \text{Is That a Factorization?} \]

\[ e^{\mathbf{y} \cdot \mathbf{x}} = e^{\mathbf{x} \cdot \mathbf{y}} \prod_{m=0}^{\infty} \frac{1}{m!} (\mathbf{y} - \mathbf{x}) \cdot \mathbf{x} \]

---

Scalar Product in d-Dimensional Space

**Definition of scalar product:**

\[ \mathbf{a} = (a_1, \ldots, a_d), \quad \mathbf{b} = (b_1, \ldots, b_d). \]

\[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \ldots + a_d b_d = \sum_{i=1}^{d} a_i b_i. \]

\[ |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}. \]

**What if**

\[ a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{C} \]

**Definition:**

\[ \mathbf{a} \cdot \mathbf{b} = \bar{a}_1 \bar{b}_1 + \ldots + \bar{a}_d \bar{b}_d = \sum_{i=1}^{d} \bar{a}_i \bar{b}_i. \]

---

Properties of Scalar Product

**Commutativity:**

\[ \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \]

**Scaling:**

\[ \left( \lambda \mathbf{a} \right) \cdot \mathbf{b} = \lambda \left( \mathbf{a} \cdot \mathbf{b} \right) = \lambda (\mathbf{a} \cdot \mathbf{b}), \quad \lambda \in \mathbb{R} \]

---

Factorization of Scalar Product Powers

\[ (\mathbf{a} \cdot \mathbf{b})^n = \left( \sum_{i=1}^{d} a_i b_i \right)^n = \sum_{n=0}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} a_i a_j a_k b_i b_j b_k \]

\[ = \sum_{n=0}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} a_i a_j a_k b_i b_j b_k \]

\[ = |\mathbf{a} \otimes \mathbf{a} \otimes \ldots \otimes \mathbf{a}| \cdot |\mathbf{b} \otimes \mathbf{b} \otimes \ldots \otimes \mathbf{b}| = a^n \cdot b^n \]

\[ a^n \otimes b^n = (a \cdot b)^n = (\mathbf{a} \cdot \mathbf{a})^n = \mathbf{b}^n \cdot \mathbf{a}^n. \]

\[ e^{\mathbf{y} \cdot \mathbf{x}} = e^{\mathbf{x} \cdot \mathbf{y}} \prod_{m=0}^{\infty} \frac{1}{m!} (\mathbf{y} - \mathbf{x}) \cdot \mathbf{x} \]

\[ = e^{\mathbf{x} \cdot \mathbf{y}} \prod_{m=0}^{\infty} \frac{1}{m!} \mathbf{x}^m \cdot (\mathbf{y} - \mathbf{x})^m. \]
Is That Factorization?

1) Truncation:
\[
\phi(y, x) = \phi_{\text{trunc}} = \sum_{n=1}^{\infty} \frac{1}{n!} x^n \cdot (y - x)^n + \text{Residual},
\]

2) Fast summation:
\[
v_j = \sum_{n=1}^{\infty} u_n \phi(y, x) = \sum_{n=1}^{\infty} \frac{1}{n!} x^n \cdot (y - x)^n + \text{Residual},
\]
\[
v_j = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{n=1}^{\infty} \frac{1}{n!} x^n \cdot (y - x)^n \right) + \text{Residual},
\]
\[
v_j = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{n=1}^{\infty} \frac{1}{n!} x^n \right) \cdot (y - x)^n + \text{Residual},
\]
\[
v_j = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{n=1}^{\infty} \frac{1}{n!} x^n \right) \cdot (y - x)^n + \text{Residual},
\]
\[
v_j = \sum_{n=1}^{\infty} c_n \cdot (y - x)^n + \text{Residual},
\]
Yes! It is!

Example (Let’s Try To Get Explicit Forms in 2D)

\[
a = (a_1, a_2),
\]
\[
a^2 = (a_1, a_1, a_2, a_2, a_2, a_2),
\]
\[
a^3 = (a_1, a_1, a_1, a_2, a_2, a_2, a_2, a_2, a_2, a_2, a_2, a_2),
\]

The length of \(a^k\) is \(2^k\) ! This is not factorial!

In \(d\) dimensions the length of \(a^k\) is even \(d^k\)

What to do in practical problems?

Use Compression!

Compression operator:
\[
A^* = \text{Compress}(a^k)
\]

Required Property:
\[
a^* \cdot b^* = \text{Compress}(a^k) \cdot \text{Compress}(b^k).
\]

Consider \(R^2\):
\[
(a^* \cdot b^*)^k = (a_1 b_1 + a_2 b_2)^k = \text{Compress}((a_1 b_1 + a_2 b_2)^k)
\]

Let us define:
\[
A^* = \text{Compress}(a^k) = \left( a_1^{\frac{1}{n!}}, a_2^{\frac{1}{n!}}, \ldots, a_1^{\frac{1}{n!}} \right)
\]
\[
B^* = \text{Compress}(b^k) = \left( b_1^{\frac{1}{n!}}, b_2^{\frac{1}{n!}}, \ldots, b_1^{\frac{1}{n!}} \right)
\]

Example of Fast Computation

\[
v_j = \sum_{n=1}^{\infty} u_n \phi(y, x) = \sum_{n=1}^{\infty} c_n \cdot (y - x)^n + \text{Residual},
\]
\[
c_n = \frac{1}{n!} \sum_{n=1}^{\infty} u_n \text{Compress}(x^n).
\]

Equivalent to:
\[
v_j = \sum_{n=1}^{\infty} c_n \cdot \text{Compress}((y - x)^n) = \text{Residual},
\]
\[
c_n = \frac{1}{n!} \sum_{n=1}^{\infty} u_n \text{Compress}(x^n).
\]

Number of multiplications (complexity) to obtain \(v_j\):
\[
\text{Complexity} = 1 + 2 + \ldots + p = \frac{p(p + 1)}{2}.
\]
Compression Can be Performed for any Dimensionality (Example for 3D):

\[ a^* \cdot b^* = (a \cdot b)^* = (a_1 b_1 + a_2 b_2 + a_3 b_3)^* \]

\[ = \left[ (a_1 b_1 + a_2 b_2 + a_3 b_3)^* \right] = \sum_{n=0}^{n} \binom{n}{m} a_1 b_1 a_2 b_2 a_3 b_3 \]

\[ = a_1^* b_1^* a_2^* b_2^* a_3^* b_3^* \]

\[ = \sum_{n=0}^{n} \binom{n}{m} \binom{n-m}{l} a_1^{n-m-l} a_2^l a_3^m b_1^{n-m-l} b_2^l b_3^m \]

\[ = \sum_{n=0}^{n} \binom{n}{m} \binom{n}{l} a_1^{n-m-l} a_2^l a_3^m b_1^{n-m-l} b_2^l b_3^m \]

\[ = a_1^* b_1^* a_2^* b_2^* a_3^* b_3^* \]

The length of the compressed vector is \((n+1)+n+\ldots+1 = (n+1)/(n+2)/2\)

Compression Can be Performed for any Dimensionality (General Case):

\[ (a_1 + a_2 + \ldots + a_d)^* = \sum_{n=0}^{n} \binom{n}{m} (n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d} \]

\[ \binom{n}{m} (n_1, n_2, \ldots, n_d) = \frac{n!}{n_1! n_2! \ldots n_d!} \]

Multinomial coefficients

\[ \text{Compress}(a^*) = \left( a_1^*, a_2^*, a_3^*, \ldots, a_d^* \right) \]

So we have

\[ a^* \cdot b^* = \text{Compress}(a^*) \cdot \text{Compress}(b^*) \]

\[ = \sum_{n=0}^{n} \binom{n}{m} (n_1, n_2, \ldots, n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d} \]

\[ = (a_1 b_1 + a_2 b_2 + \ldots + a_d b_d)^* = (a \cdot b)^* \]

What are multinomial coefficients?

\((n : n_1, n_2, \ldots, n_d)\) is the number of ways of putting \(n\) different objects into \(d\) different boxes with \(n_k\) in the \(k\)-th box.

The length of the compressed vector

\[ d = 1 : \frac{1}{n+1}, \quad d = 2 : \frac{n+1}{n+2}, \quad d = 3 : \frac{1}{2(n+1)(n+2)} \]

Theorem: If \(a \in \mathbb{R}^d\), then the length of the compressed vector \(\text{Compress}(a^*)\) is

\[ \frac{n+d-1}{n} \binom{n+d-1}{d} = \frac{(n+1)\ldots(n+d-1)}{(d-1)!} \]

Proof: We have a base for induction (see above). Let this hold for \(d\) dimensions. Consider \(d+1\) dimensions:

\[ ((n_1 + \ldots + n_d) + a_{d+1})^* \]

\[ = \sum_{n=0}^{n} \binom{n}{m} (n_1 + \ldots + n_d) a_1^{n_1} a_2^{n_2} \ldots a_d^{n_d} a_{d+1}^{n_d} \]

The number of terms is then

\[ \sum \binom{n+d-1}{m} \binom{d-1}{m-1} \binom{d}{n} \binom{n+d-1}{n} = \binom{n+d}{n} \]

This proves the theorem.
Example of Fast Computation

\[ v_y = \sum_{y=0}^{M} y \cdot (y - x)^t + \text{Residual}, \quad c_y = \frac{1}{m} \sum_{y=0}^{M} y \cdot e^{ax} \cdot y^t. \]

Equivalent to:

\[ v_y = \sum_{y=0}^{M} c_y \cdot \text{Compress}
\left( (y - x)^t \right) + \text{Residual}, \quad c_y = \frac{1}{m} \sum_{y=0}^{M} y \cdot e^{ax} \cdot \text{Compress}(y^t). \]

Number of multiplications (complexity) to obtain \( v_y \) (in 2D case!)

\[ \text{Complexity} = 1 + 2 + \ldots + p = \frac{p(p+1)}{2}. \]

\[ c_1 = \sum_{y=0}^{M} y \cdot e^{ax}, \]

\[ c_2 = \langle c_1, c_2 \rangle = \sum_{y=0}^{M} y \cdot e^{ax} \cdot (x_1, x_2), \]

\[ c_3 = \langle c_1, c_2, c_3 \rangle = \sum_{y=0}^{M} y \cdot e^{ax} \cdot \left( x_1^2, x_1 x_2, x_2^2 \right). \]

## Complexity of Fast Summation

Let \( \phi \) be a scalar product of vectors \( A_i \) and \( F_i \) of length \( P(p) \) (\( p \) is the truncation number).

- Complexity of summation over \( i \) is then \( O(P_N) \).
- Complexity of scalar product operation is \( P \).
- Complexity of \( M \) scalar product operations is \( O(P_M) \) (for \( j = 1, \ldots, M \)).

Total complexity is \( O(P_M + P_N) \).

Fast Method is more efficient than direct only if \( O(P_M + P_N) < O(MN) \), so we should have

\[ P(p) \ll \min(M, N) \]

## General Forms of Factorization for Fast Summation (1)

\[ v_y = \sum_{i=1}^{N} a_i \cdot \Phi(y, x), \quad f = 1, \ldots, M. \]

\[ \Phi(y, x) = \sum_{i=1}^{N} a_i (x, x) \cdot \langle y - x, y - x \rangle = \text{Error}(x, x, y). \]

How about vectors of length \( p \)

\[ v_y = \sum_{i=1}^{N} a_i \cdot \Phi_i(y), \quad \Phi_i(y) \text{ Some parameter depending on } i \]

More general to have

\[ v_y = \sum_{i=1}^{N} a_i \cdot \Phi_i(y) \quad \text{or} \quad v(y) = \sum_{i=1}^{N} A_i \cdot x, \]

## General Forms of Factorization for Fast Summation (2)

The potential can be factorized as

\[ \Phi(y) = A(x) \cdot F(y - x), \]

Generalized product \( \cdot \) can be scalar product, contraction, etc. \( A_i \) and \( F \) can be real or complex vectors, tensors, etc. in \( p \)-dimensional space.

Requirements to the product (distributivity with respect to addition)

\[ (\alpha A + \beta A) = \alpha A + \beta A = F. \]

In this case

\[ v(y) = \sum_{i=1}^{N} a_i \cdot \Phi_i(y) = \sum_{i=1}^{N} A_i \cdot (x) \cdot \langle y - x, y - x \rangle = A(x) \cdot F(y - x), \]

\[ A(x) = \sum_{i=1}^{N} a_i A_i(x). \]

We do not need commutativity of \( \cdot \) (i.e. we do not request \( A \cdot F = F \cdot A \)).
General Forms of Factorization for Fast Summation (3)

Actually, we even do need continuous variable $y$. The problem is to represent all matrix elements in the form

$$\Phi_j = A_i \cdot F_j$$

then

$$v_j = \sum_{i=1}^{N} u_i \Phi_j = \sum_{i=1}^{N} u_i (A_i \cdot F_j) = \left( \sum_{i=1}^{N} u_i A_i \right) \cdot F_j.$$

Outline

- Far Field Expansions (or S-expansions)
  - Regular Potential (Convergent Series);
  - Regular Potential (Asymptotic Series);
  - Singular Potential;
- Asymptotic Series
- Approaches for Selection of the Basis Functions

Far Field Expansions (S-expansions)

Let $x_i \in \mathbb{R}^d$.

We call expansion

$$\Psi(y, x_i) = \sum_{i=0}^{\infty} b_{i0}(x_i) S_i(y - x_i)$$

far field expansion (or S-expansion) outside a sphere

$$|y - x_i| > R_*,$$

if the series converges for $\forall y, |y - x_i| > R_*$. Might be Singular (at $y = x_i$) Basis Functions

Far Field Expansion of a Regular Potential

...sometimes like this:

Can be like this:

$$|y - x_i| > R, |x - x_i| > R_*$$

...sometimes like this:

$$|y - x_i| > R, |x - x_i| > R_*$$
Local Expansion of a Regular Potential
Can be Far Field Expansion Also

Valid for any \( r_* < \infty \), and \( x_i \)

\[
\Phi(y, x_i) = e^{-r_\ast y^2} = \sum_{m=0}^{\infty} a_m(x_i, x_i) S_m(y - x_i).
\]

We have

\[
e^{-r_\ast y^2} = \sum_{m=0}^{\infty} S_m(x_i - x_\ast) R_m(y - x_\ast) = \sum_{m=0}^{\infty} R_m(x_i - x_\ast) S_m(y - x_\ast)
\]

\[
R_m(x) = S_m(x) = \sqrt{\frac{2m}{m!}} e^{-x^2} x^m, \quad m = 0, 1, \ldots
\]

Or may be not…

Decays at large \( y \)

\[
e^{-r_\ast y^2} = \sum_{l=0}^{\infty} h_l(x_i - x_\ast) \left(\frac{y - x_\ast}{l!}\right)^l = \sum_{l=0}^{\infty} \frac{(y - x_\ast)^l}{l!} h_l(y - x_\ast)
\]

Local Far

Asymptotic Series

\[
f(x, \epsilon) = f_0(x) \phi_0(\epsilon) + f_1(x) \phi_1(\epsilon) + f_2(x) \phi_2(\epsilon) + \ldots = \sum_{n=0}^{\infty} f_n(x) \phi_n(\epsilon)
\]

\[
\lim_{\epsilon \to 0} \phi_n(\epsilon) = 0, \quad \text{Gauge functions}
\]

\[
f(x, \epsilon) - \sum_{n=0}^{N-1} f_n(x) \phi_n(\epsilon) = O(\phi_n(\epsilon))
\]

The asymptotic expansion is uniform in domain \( x \in \Omega \) if

\[\forall x \in \Omega, \quad \left| f(x, \epsilon) - \sum_{n=0}^{N-1} f_n(x) \phi_n(\epsilon) \right| = O(\phi_n(\epsilon)).\]

Otherwise the asymptotic expansion is not uniform.

Examples of Uniform and Non-Uniform Expansions

Example of uniform expansion:

\[
f(x, \epsilon) = \frac{1}{x + \epsilon}, \quad x > 10
\]

\[
f(x, \epsilon) = \frac{1}{x + \epsilon} = \frac{1}{x} \left(1 + \frac{\epsilon}{x}\right)^{-1} = \frac{1}{x} \left(1 - \frac{1}{2} \epsilon x^{-1} + O(\epsilon^2 x^{-2})\right)
\]

Example of non-uniform expansion:

\[
f(x, \epsilon) = \epsilon^x, \quad x \in \mathbb{R}^1
\]

\[
\epsilon^x = \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!}
\]

Prove that! (Hint: consider \( x \gg \epsilon^{-1} \)).
Example of Far Field Expansion of a Regular Function (Using Asymptotic Series)

\[ \Phi(y, x) = \frac{1}{1 + (y - x)^2} = \frac{1}{1 + (y - x_0 + (x_0 - x))^2} = \frac{1}{(y - x_0)^2 + (x - x_0)^2} \]

Let

\[ \epsilon = \frac{1}{y - x_0} \]

\[ \Phi(y, x, x_0) = \epsilon^2 \frac{1}{1 + \left[ \frac{1}{2 \epsilon^2} (x - x_0)^2 \right]} = \epsilon^2 \left( 1 + \epsilon^2 (x - x_0)^2 \right)^{\epsilon^2} \]

\[ f(x, \epsilon) = \frac{1}{\epsilon^2 (1 - \epsilon^2)} = \sum_{n=0}^{\infty} f_n(x) \epsilon^n \]

\[ f_0(x) = \frac{1}{\epsilon^2} \frac{\partial f(x, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} \]

Example of Far Field Expansion of a Regular Function (continuation)

\[ f_0(x) = 1, \]

\[ f_1(x) = \frac{\partial f(x, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} = 2x, \]

\[ f_2(x) = \frac{1}{2} \frac{\partial^2 f(x, \epsilon)}{\partial \epsilon^2} \bigg|_{\epsilon=0} = 3x^2 - 1, \]

\[ \Phi(y, x, x_0) = \frac{1}{(y - x_0)^2} \sum_{n=0}^{\infty} f_n(x - x_0) \frac{1}{(y - x_0)^2} \]

\[ y > 100, \quad x_0 = 1, \quad x = 0, \quad \epsilon = 10^{-3}, \quad x = 1 \]

\[ \Phi(y, x, x_0) \sim \frac{1}{(y - x_0)^2} \left( 1 + \frac{2(y - x_0)}{(y - x_0)^2} \right) \sim e^{\epsilon^2 (x^2 - 1)} = 2 \cdot 10^{-9}. \]

Far Field Expansion of a Singular Potential

...sometimes like this:

\[ |y - x| > R, \quad |x - x_0| \]

...sometimes like this:

Can be like this:

\[ |y - x| > R, \quad |x - x_0| \]

This case only!

Example For S-expansion of Singular Potential

\[ \Phi(y, x, x_0) = \frac{1}{y - x_0}, \]

\[ \frac{1}{y - x_0} = \frac{1}{y - x_0} \left( 1 - \frac{x_0 - x}{y - x_0} \right) \frac{1}{(y - x_0) \left( \left[ 1 - \frac{x_0 - x}{y - x_0} \right] \right)^m} \]

\[ 1 - \frac{x_0 - x}{y - x_0} \]

\[ \sum_{m=0}^{\infty} b_m(x - x_0) S_m(y - x_0), \quad |y - x_0| > |x - x_0| \]

\[ \Phi(y, x, x_0) = \sum_{m=0}^{\infty} b_m(x - x_0)^m S_m(y - x_0), \quad m = 0, 1, \ldots \]

\[ b_m(x, y) = (x - x_0)^m, \quad m = 0, 1, \ldots \]

\[ S_m(x) = (x - x_0)^{m-1}, \quad m = 0, 1, \ldots \]
Let us compare with the R-expansion of the same function

\[ |y - x| < |y - x^*| : \]
\[ \Phi(y, x^*) = \sum_{m=0}^{\infty} a_m(x, x^*) R_m(y - x^*), \]
\[ a_m(x, x^*) = \frac{(y - x^*)^{-m-1}}{m+1}, \quad m = 0, 1, \ldots, \]
\[ R_m(y - x^*) = (y - x^*)^m, \quad m = 0, 1, \ldots \]

\[ |y - x| > |y - x^*| : \]
\[ \Phi(y, x^*) = \sum_{m=0}^{\infty} b_m(x, x^*) S_m(y - x^*), \]
\[ b_m(x, x^*) = \frac{(y - x^*)^{-m-1}}{m+1}, \quad m = 0, 1, \ldots, \]
\[ S_m(y - x^*) = (y - x^*)^m, \quad m = 0, 1, \ldots \]

Singular Point is located at the Boundary of regions for the R- and S-expansions!

What Do We Need For Real FMM (that provides spatial grouping)

We need S-expansion for \( |y - x| > R > |x - x^*| \)
We need R-expansion for \( |y - x| < r < |x - x^*| \)

Basis Functions

- Power series are great, but do they provide the best approximation? (sometimes yes!)
- Other approaches to factorization:
  - Asymptotic Series (Can be divergent!);
  - Orthogonal Bases in \( L^2 \);
  - Eigen Functions of Differential Operators;
  - Functions Generated by Differentiation or Other Linear Operators.
- Some of this approaches will be considered in this course.