

MAIT 627 Fast Multipole Methods

Lecture 3

Outline

- **Factorization – One of key parts of the FMM**
 - Extensions of our trick for fast summation
 - “Middleman” scheme
 - Singular and regular fields
 - Far field and near field
- **Local Expansions (or R-expansions)**
 - Local expansions of regular and singular potentials
 - Power series
 - Taylor series
- **Far Field Expansions (or S-expansions)**
 - Far field expansions of regular and singular potentials
 - Asymptotic series

Matrix-Vector Multiplication

Compute matrix vector product

$$v = \Phi u$$

or

$$v_j = \sum_{i=1}^N \Phi_{ji} u_i, \quad j = 1, \dots, M,$$

where

$$\Phi_{ji} = \Phi(y_j, x_i), \quad j = 1, \dots, M, \quad i = 1, \dots, N,$$

or

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2N} \\ \dots & \dots & \dots & \dots \\ \Phi_{M1} & \Phi_{M2} & \dots & \Phi_{MN} \end{pmatrix} = \begin{pmatrix} \Phi(y_1, x_1) & \Phi(y_1, x_2) & \dots & \Phi(y_1, x_N) \\ \Phi(y_2, x_1) & \Phi(y_2, x_2) & \dots & \Phi(y_2, x_N) \\ \dots & \dots & \dots & \dots \\ \Phi(y_M, x_1) & \Phi(y_M, x_2) & \dots & \Phi(y_M, x_N) \end{pmatrix}$$

Generally we have two sets of points in d -dimensions:

$$\text{Sources: } X = \{x_1, \dots, x_N\}, \quad x_i \in \mathbb{R}^d, \quad i = 1, \dots, N,$$

$$\text{Receivers: } Y = \{y_1, \dots, y_M\}, \quad y_j \in \mathbb{R}^d, \quad j = 1, \dots, M,$$

The receivers also can be called “targets” or “evaluation points”.

Why \mathbb{R}^d ?

- $d = 1$
 - Scalar functions, interpolation, etc.
- $d = 2, 3$
 - Physical problems in 2 and 3 dimensional space
- $d = 4$
 - 3D Space + time, 3D grayscale images
- $d = 5$
 - Color 2D images, Motion of 3D grayscale images
- $d = 6$
 - Color 3D images
- $d = 7$
 - Motion of 3D color images
- $d = \text{arbitrary}$
 - d -parametric spaces, statistics, database search procedures

Fields (Potentials)

Field (Potential) of a single
(i th) unit source

$$\begin{aligned} \phi(\mathbf{y}) &= \sum_{i=1}^M u_i \Phi(\mathbf{y}, \mathbf{x}_i), \quad \mathbf{y} \in \mathbb{R}^d, \\ \mathbf{v}_j &= \mathbf{v}(\mathbf{y}_j), \quad j = 1, \dots, M. \end{aligned}$$

Field (Potential) of the set
of sources of intensities $\{u_i\}$

Fields are continuous!
(Almost everywhere)

Examples of Fields

- There can be vector or scalar fields (we focus mostly on scalar fields)
- Fields can be *regular* or *singular*

Scalar Fields:

Gravity

(singular at $\mathbf{y} = \mathbf{x}_i$)

$$\Phi(\mathbf{y}, \mathbf{x}_i) = \frac{1}{|\mathbf{y} - \mathbf{x}_i|}$$

Monochromatic Wave (k is the wavenumber)

(singular at $\mathbf{y} = \mathbf{x}_i$)

$$\Phi(\mathbf{y}, \mathbf{x}_i) = \frac{\exp\{ik|\mathbf{y} - \mathbf{x}_i|\}}{|\mathbf{y} - \mathbf{x}_i|}$$

Gaussian

(regular everywhere)

$$\Phi(\mathbf{y}, \mathbf{x}_i) = \exp\{-|\mathbf{y} - \mathbf{x}_i|^2/\sigma\}$$

Vector Field:

3D Velocity field:

(singular at $\mathbf{y} = \mathbf{x}_i$)

$$\begin{aligned} \Phi(\mathbf{y}, \mathbf{x}_i) &= \nabla_{\mathbf{y}} \frac{1}{|\mathbf{y} - \mathbf{x}_i|} = i_1 \frac{\partial}{\partial y_1} \frac{1}{|\mathbf{y} - \mathbf{x}_i|} + i_2 \frac{\partial}{\partial y_2} \frac{1}{|\mathbf{y} - \mathbf{x}_i|} + i_3 \frac{\partial}{\partial y_3} \frac{1}{|\mathbf{y} - \mathbf{x}_i|}, \\ \mathbf{y} &= (y_1, y_2, y_3) \in \mathbb{R}^3. \end{aligned}$$

Straightforward Computational Complexity:

$$O(MN) \quad \text{Error: 0 ("machine" precision)}$$

The Fast Multipole Methods look for computation of the same problem with complexity $\alpha(MN)$ and error $<$ prescribed error.

In the case when the error of the FMM does not exceed the machine precision error (for given number of bits) there is no difference between the "exact" and "approximate" solution.

Factorization "Middleman Method"

Global Factorization

$\forall \mathbf{x}_i, \mathbf{y}_j \in \Omega \subset \mathbb{R}^d :$

$$\Phi(\mathbf{y}_j, \mathbf{x}_i) = \sum_{m=0}^{\infty} a_m(\mathbf{x}_i - \mathbf{x}_*) f_m(\mathbf{y}_j - \mathbf{x}_*) = \sum_{m=0}^{p-1} a_m(\mathbf{x}_i - \mathbf{x}_*) f_m(\mathbf{y}_j - \mathbf{x}_*) + \text{Error}(p, \mathbf{x}_i, \mathbf{y}_j)$$

Expansion coefficients Expansion center Truncation number Basis functions

Factorization Trick

$$\begin{aligned} v_j &= \sum_{i=1}^N \Phi(\mathbf{y}_j, \mathbf{x}_i) u_i \\ &= \sum_{i=1}^N \left[\sum_{m=0}^{p-1} a_m(\mathbf{x}_i - \mathbf{x}_*) f_m(\mathbf{y}_j - \mathbf{x}_*) + \text{Error}(p, \mathbf{x}_i, \mathbf{y}_j) \right] u_i \\ &= \sum_{m=0}^{p-1} f_m(\mathbf{y}_j - \mathbf{x}_*) \sum_{i=1}^N a_m(\mathbf{x}_i - \mathbf{x}_*) u_i + \sum_{i=1}^N \text{Error}(p, \mathbf{x}_i, \mathbf{y}_j) u_i \\ &= \sum_{m=0}^{p-1} c_m f_m(\mathbf{y}_j - \mathbf{x}_*) + \text{Error}(N, p), \end{aligned}$$

where

$$c_m = \sum_{i=1}^N a_m(\mathbf{x}_i - \mathbf{x}_*) u_i.$$

Reduction of Complexity

Straightforward (nested loops):

```
for j = 1, ..., M
  v_j = 0;
  for i = 1, ..., N
    v_j = v_j + Phi(y_j, x_i) u_i;
  end;
end;
```

Complexity: $O(MN)$

Factorized:

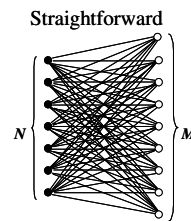
```
for m = 0, ..., p-1
  c_m = 0;
  for i = 1, ..., N
    c_m = c_m + a_m(x_i - x_*) u_i;
  end;
end;

for j = 1, ..., M
  v_j = 0;
  for m = 0, ..., p-1
    v_j = v_j + c_m f_m(y_j - x_*);
  end;
end;
```

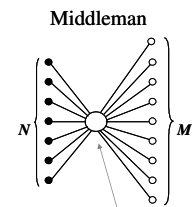
Complexity: $O(pN+pM)$

If $p \ll \min(M, N)$ then complexity reduces!

Middleman Scheme



Complexity: $O(pN+pM)$



Set of coefficients $\{c_m\}$

Far Field and Near Field

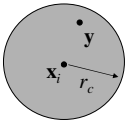
■ Near Field of the i th source:

$$|y - x_i| < r_c$$

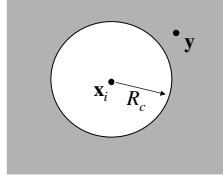
■ Far Field of the i th source:

$$|y - x_i| > R_c$$

Near Field



Far Field



What are these r_c and R_c ?

depends on the potential + some conventions for the terminology

Local (Regular) Expansion

Do not confuse with the Near Field!

Let

$$x_i \in \mathbb{R}^d$$

Basis Functions

We call expansion

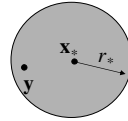
$$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_i) R_m(y - x_i)$$

local (regular) inside a sphere

$$|y - x_i| < r_*$$

Expansion Coefficients

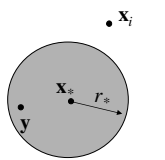
if the series converges for $\forall y, |y - x_i| < r_*$.



We also call this **R-expansion**, since basis functions R_m should be *regular*

Local Expansion of a Regular Potential

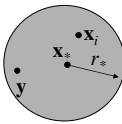
Can be like this:



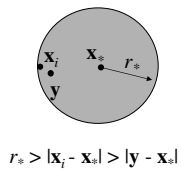
$$|y - x_s| < r_s < |x_i - x_s|$$

...or like this:

...or like this:



$$r_s > |y - x_s| > |x_i - x_s|$$



$$r_s > |x_i - x_s| > |y - x_s|$$

Local Expansion of a Regular Potential (Example)

Valid for any $r_s < \infty$, and x_i

$$x, y \in \mathbb{R}^1$$

$$\Phi(y, x_i) = e^{-\vartheta|y-x_i|}$$

Looking for factorization:

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i - x_s) R_m(y - x_s)$$

We have

$$\begin{aligned} e^{-\vartheta|y-x_i|} &= e^{-\vartheta|x_i-x_s| - \vartheta(y-x_s)} = e^{-\vartheta|x_i-x_s|} e^{-\vartheta(y-x_s)} \\ &= e^{-\vartheta|x_i-x_s|} e^{-\vartheta(y-x_s)} \sum_{m=0}^{\infty} \frac{2^m (x_i - x_s)^m (y - x_s)^m}{m!} \end{aligned}$$

Choose

$$\begin{aligned} a_m(x_i - x_s) &= e^{-\vartheta|x_i-x_s|} \sqrt{\frac{2^m}{m!}} (x_i - x_s)^m, \quad m = 0, 1, \dots \\ R_m(y - x_s) &= e^{-\vartheta|y-x_s|} \sqrt{\frac{2^m}{m!}} (y - x_s)^m, \quad m = 0, 1, \dots \end{aligned}$$

Local Expansion of a Regular Potential (The same kernel, Example 2)

$$e^{-(y-x_*)^2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (y-x_*)^{2m}$$

So

$$e^{-(y-x_i)^2} = e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} = \sum_{m=0}^{\infty} \frac{2^m (x_i-x_*)^m (y-x_*)^m}{m!} = e^{-(x_i-x_*)^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n 2^m (x_i-x_*)^m (y-x_*)^{m+2n}}{m!n!}$$

Rearrange summation:

$$m+2n=l$$

$$m=l-2n$$

$$e^{-(y-x_i)^2} = e^{-(x_i-x_*)^2} \sum_{l=0}^{\infty} \sum_{n=0}^{\lfloor l/2 \rfloor} \frac{(-1)^n 2^{l-2n} (x_i-x_*)^{l-2n}}{(l-2n)!n!} (y-x_*)^l = \sum_{l=0}^{\infty} h_l(x_i-x_*) \frac{(y-x_*)^l}{l!}$$

Hermit polynomials:

$$H_l(x) = l! \sum_{n=0}^{\lfloor l/2 \rfloor} \frac{(-1)^n 2^{l-2n} (x)^{l-2n}}{(l-2n)!n!}$$

Hermit functions:

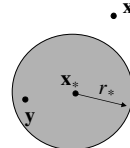
$$h_l(x) = e^{-x^2} H_l(x)$$

Choose

$$a_l(x_i-x_*) = h_l(x_i-x_*), \quad R_l(y-x_*) = \frac{1}{l!} (y-x_*)^l, \quad l=0,1,\dots$$

Local Expansion of a Singular Potential

Can be like this:

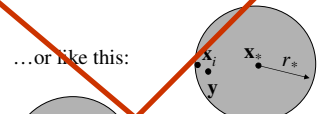


$$|y-x_*| < r_* \leq |x_i-x_*|$$

Like this only!

...or like this:

...or like this:



$$r_* > |x_i-x_*| > |y-x_*|$$

$$r_* > |y-x_*| > |x_i-x_*|$$

Never ever!

Because x_i is a singular point!

Local Expansion of a Singular Potential (Example)

Valid for any $|x_i-x_*| > |y-x_*|$

$$x, y \in \mathbb{R}^1$$

$$\Phi(y, x_i) = \frac{1}{y-x_i}$$

Looking for factorization:

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i-x_*) R_m(y-x_*)$$

We have

$$\frac{1}{y-x_i} = \frac{1}{y-x_* - (x_i-x_*)} = \frac{1}{(y-x_*) [1 - \frac{x_i-x_*}{y-x_*}]} = \frac{1}{(y-x_*)} \left[1 - \frac{x_i-x_*}{y-x_*} \right]^{-1}$$

Geometric progression:

$$(1-a)^{-1} = 1 + a + a^2 + \dots = \sum_{m=0}^{\infty} a^m, \quad |a| < 1$$

$$\left[1 - \frac{x_i-x_*}{y-x_*} \right]^{-1} = \sum_{m=0}^{\infty} \frac{(y-x_*)^m}{(x_i-x_*)^m}, \quad |y-x_*| < |x_i-x_*|$$

Choose

$$a_m(x_i-x_*) = -\frac{1}{(x_i-x_*)^{m+1}}, \quad m=0,1,\dots$$

$$R_m(y-x_*) = (y-x_*)^m, \quad m=0,1,\dots$$

Power and Taylor Series

- Power and Taylor Series
 - Power Series in 1D
 - Taylor Series in 1D
- Multidimensional Taylor Series
- Factorization of Scalar Products in \mathbb{R}^d
- Compression of Factorized Series
- Factorization of Scalar Products in \mathbb{R}^d (compression)
 - Factorization in 2D.
 - Factorization in 3D.
 - Factorization in dD .
 - Multinomial Coefficients.
 - Complexity of Fast Summation.
- General Forms of Factorization for Fast Summation

Power Series

Power series relative to real or complex variable y is a series of type

$$f(y - x_*) = \sum_{m=0}^{\infty} a_m (y - x_*)^m,$$

where a_m are real or complex numbers.

Properties of Power Series

1) For any power series there exists r_* , such that the series converges absolutely at $|y - x_*| < r_*$, and diverges at $|y - x_*| > r_*$. The number r_* is called *the convergence radius* of the series, $0 \leq r_* \leq \infty$.

For any number q , such that $0 < q < r_*$, the power series uniformly converges at $|y - x_*| < q$.

Properties of Power Series

2) Convergent power series can be summed, multiplied by a scalar, or multiplied according to the Cauchy rule.

For $|y - x_*| < r_*$, the sum of the series is a continuous and infinitely differentiable function of y .

The power series can be differentiated term by term at $|y - x_*| < r_*$, and integrated over any closed interval included in $|y - x_*| < r_*$.

Differentiated or integrated series (if integration is taken from x_* to $y - x_*$) have the same convergence radius r_* .

$$\sum_{m=0}^{\infty} a_m (y - x_*)^m + \sum_{m=0}^{\infty} b_m (y - x_*)^m = \sum_{m=0}^{\infty} (a_m + b_m) (y - x_*)^m,$$

$$\alpha \sum_{m=0}^{\infty} a_m (y - x_*)^m = \sum_{m=0}^{\infty} \alpha a_m (y - x_*)^m,$$

$$\left[\sum_{m=0}^{\infty} a_m (y - x_*)^m \right] \left[\sum_{m=0}^{\infty} b_m (y - x_*)^m \right] = \sum_{n=0}^{\infty} \left[\sum_{m=0}^n a_m b_{n-m} \right] (y - x_*)^n.$$

Cauchy's rule →

Properties of Power Series

3) Uniqueness. If there exists such positive r that at any y satisfying $|y - x_*| < r$ two power series have the same sum, then the coefficients of these series are the same.

For those who love proofs

Prove the above properties!

(Not the course formal requirement, but a good exercise)

Taylor Series (Finite)

Let $f(y)$ be a real function, $f(y) \in D^n(x_*, x_* + r_*)$ (so the n -th derivative $f^{(n)}(y)$ exists for $x_* < y < x_* + r_*$). Then

$$f(y) = f(x_*) + f'(x_*)(y-x_*) + \frac{1}{2!}f''(x_*)(y-x_*)^2 + \dots + \frac{1}{(n-1)!}f^{(n-1)}(x_*)(y-x_*)^{n-1} + \text{Residual}_n(y).$$

Cauchy's evaluation:

$$|\text{Residual}_n(y)| \leq \frac{|y-x_*|^n}{n!} \sup_{x_* \leq \xi \leq x_* + r_*} |f^{(n)}(\xi)|$$

Lagrange evaluation:

$$\text{Residual}_n(y) = \int_{x_*}^y dx \int_{x_*}^x dx \dots \int_{x_*}^x f^{(n)}(x) dx = \frac{1}{n!} f^{(n)}(\xi)(y-x_*)^n, \\ \xi \in (x_*, x_* + r_*).$$

We have similar formulae for $x_* - r_* < y < x_*$.

Taylor Series (Infinite)

Let $f(y) \in D^\infty(x_* - r_*, x_* + r_*)$ and let

$$\lim_{n \rightarrow \infty} \text{Residual}_n(y) = 0,$$

then

$$f(y) = \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(x_*) (y-x_*)^m, \quad |y-x_*| < r_*,$$

and the series uniformly converges to $f(y)$ for any $|y-x_*| \leq q$, where $0 \leq q < r_*$.

Local 1D Taylor Expansion

Looking for local expansion:

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y-x_*),$$

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi}{\partial y^m}(x_*, x_i) (y-x_*)^m.$$

$$a_m(x_i, x_*) = \frac{1}{m!} \frac{\partial^m \Phi}{\partial y^m}(x_*, x_i), \quad m = 0, 1, \dots$$

$$R_m(y-x_*) = (y-x_*)^m, \quad m = 0, 1, \dots$$

Local 1D Taylor Expansion (Example)

$$\Phi(y, x_1) = e^{y x_1}$$

$$\frac{\partial^m \Phi}{\partial y^m}(y, x_1) = x_1^m e^{y x_1}, \quad \frac{\partial^m \Phi}{\partial y^m}(x_1, x_1) = x_1^m e^{x_1^2}$$

$$a_m(x_1, x_1) = \frac{1}{m!} \frac{\partial^m \Phi}{\partial y^m}(x_1, x_1) = \frac{x_1^m}{m!} e^{x_1^2}$$

$$\Phi(y, x_1) = e^{y x_1} = \sum_{m=0}^{\infty} \frac{x_1^m}{m!} (y - x_1)^m$$

Residual for $|y - x_1| < \alpha$ (assume $x_1 > 0, x_1 \geq 0$):

$$|\text{Residual}(y)| \leq \frac{|y - x_1|^m}{m!} \sup_{x_1 = \Phi(x_1, x_1)} \left| \frac{\partial^m \Phi}{\partial y^m}(y, x_1) \right| < \frac{\alpha^m x_1^m e^{x_1^2 (1 + \alpha)}}{m!}$$

For $n = 5, \alpha = 0.5, x_1 = 1, x_1 = 0.5$ we have

$$|\text{Residual}(y)| < \frac{e}{2^2 5!} < \frac{3}{32 \cdot 120} = \frac{1}{1280} < 10^{-3}$$

Multidimensional Taylor Series

Let $f(\mathbf{y})$ be a real function.

$$f(\mathbf{y}) \in \mathcal{D}^m(\mathbf{U}_x), \quad \mathbf{y} = (y_1, \dots, y_d) \in \mathbf{U}_x \subset \mathbb{R}^d, \quad \mathbf{x}_* = (x_1, \dots, x_d) \in \mathbb{R}^d$$

Then we can write

$$f(\mathbf{y}) = f(y_1, y_2, \dots, y_d)$$

$$\begin{aligned} f(y_1, y_2, \dots, y_d) &= \sum_{m_1=0}^{\infty} \frac{1}{m_1!} \frac{\partial^{m_1} f}{\partial y_1^{m_1}}(x_1, y_2, \dots, y_d) (y_1 - x_1)^{m_1} \\ &= \sum_{m_1=0}^{\infty} \frac{1}{m_1!} \sum_{m_2=0}^{\infty} \frac{1}{m_2!} \frac{\partial^{m_1}}{\partial y_1^{m_1}} \frac{\partial^{m_2}}{\partial y_2^{m_2}} f(x_1, x_2, \dots, y_d) (y_1 - x_1)^{m_1} (y_2 - x_2)^{m_2} \\ &= \dots \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dots \sum_{m_d=0}^{\infty} \frac{\partial^{m_1}}{\partial y_1^{m_1}} \frac{\partial^{m_2}}{\partial y_2^{m_2}} \dots \frac{\partial^{m_d}}{\partial y_d^{m_d}} f(x_1, x_2, \dots, x_d) \prod_{i=1}^d \frac{1}{m_i!} (y_i - x_i)^{m_i} \end{aligned}$$

Multidimensional Taylor Series (using some vector algebra)

Operator ∇ :

$$\nabla = \mathbf{i}_1 \frac{\partial}{\partial y_1} + \dots + \mathbf{i}_d \frac{\partial}{\partial y_d}$$

Differential along direction \mathbf{s} :

$$\frac{d^m f(\mathbf{y})}{ds^m} = (\mathbf{s} \cdot \nabla)^m f(\mathbf{y}), \quad |\mathbf{s}| = 1$$

Taylor series (let $\mathbf{s} = (\mathbf{y} - \mathbf{x}_*) / |\mathbf{y} - \mathbf{x}_*|$)

$$\begin{aligned} f(\mathbf{y}) &= f(\mathbf{x}_*) + \frac{df(\mathbf{x}_*)}{ds} |\mathbf{y} - \mathbf{x}_*| + \frac{1}{2!} \frac{d^2 f(\mathbf{x}_*)}{ds^2} |\mathbf{y} - \mathbf{x}_*|^2 + \dots \\ &= f(\mathbf{x}_*) + [(\mathbf{y} - \mathbf{x}_*) \cdot \nabla] f(\mathbf{x}_*) + \frac{1}{2!} [(\mathbf{y} - \mathbf{x}_*) \cdot \nabla]^2 f(\mathbf{x}_*) + \dots \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \nabla]^m f(\mathbf{x}_*) \end{aligned}$$

Example

$$\Phi(\mathbf{y}, \mathbf{x}_1) = e^{y^{\mathbf{x}_1}} = \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \nabla_{\mathbf{x}_1}]^m \Phi(\mathbf{x}_*, \mathbf{x}_1)$$

Fix $(\mathbf{y} - \mathbf{x}_*)$:

$$\Phi(\mathbf{x}_*, \mathbf{x}_1) = e^{x_*^{\mathbf{x}_1}}$$

$$\nabla_{\mathbf{x}_1} \Phi(\mathbf{x}_*, \mathbf{x}_1) = \mathbf{x}_1 e^{x_*^{\mathbf{x}_1}} = \mathbf{x}_1 \Phi(\mathbf{x}_*, \mathbf{x}_1)$$

$$[(\mathbf{y} - \mathbf{x}_*) \cdot \nabla_{\mathbf{x}_1}] \Phi(\mathbf{x}_*, \mathbf{x}_1) = [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_1] \Phi(\mathbf{x}_*, \mathbf{x}_1)$$

$$[(\mathbf{y} - \mathbf{x}_*) \cdot \nabla_{\mathbf{x}_1}]^m \Phi(\mathbf{x}_*, \mathbf{x}_1) = [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_1]^m \Phi(\mathbf{x}_*, \mathbf{x}_1)$$

$$\Phi(\mathbf{y}, \mathbf{x}_1) = \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_1]^m \Phi(\mathbf{x}_*, \mathbf{x}_1) = e^{x_*^{\mathbf{x}_1}} \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_1]^m$$

$$\text{Check: } e^{y^{\mathbf{x}_1}} = e^{x_*^{\mathbf{x}_1}} e^{(y - x_*)^{\mathbf{x}_1}} = e^{x_*^{\mathbf{x}_1}} \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_1]^m$$