FMM CMSC 878R/AMSC 698R

Lecture 21

Outline

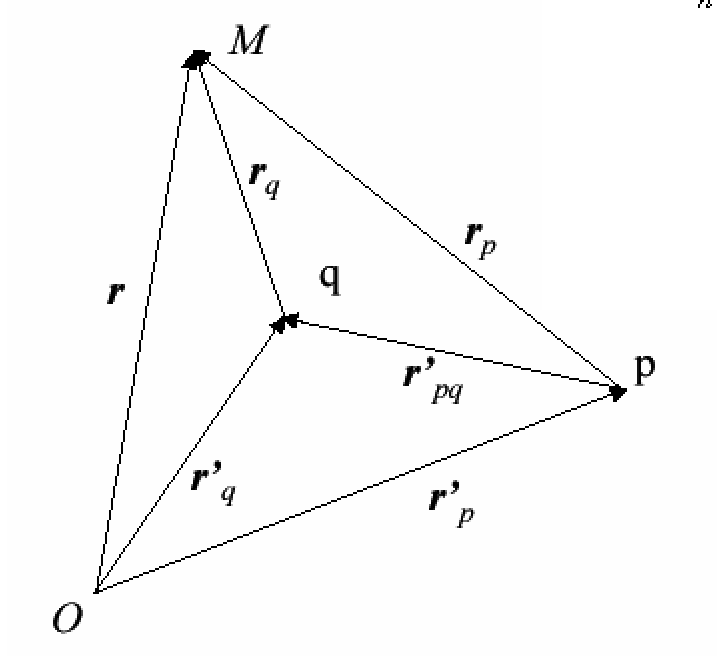
- Translation of Elementary Solutions for 3D Laplace Equation
- Differentiation of Elementary Solutions
- Recursive Computation of Translation Matrices
- Rotation of Elementary Solutions
- Decomposition of Translation into Rotation and Coaxial Translation Operations

Translations of elementary solutions of the 3D Laplace equation

$$S_n^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(S|R)_{ln}^{sm}(\mathbf{r}'_{pq})R_l^s(\mathbf{r}_q), \qquad |\mathbf{r}_q| < |\mathbf{r}'_{pq}|, \quad p \neq q.$$

$$S_n^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(S|S)_{ln}^{sm}(\mathbf{r}'_{pq})S_l^s(\mathbf{r}_q), \qquad |\mathbf{r}_q| > |\mathbf{r}'_{pq}|,$$

$$R_n^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(R|R)_{ln}^{sm}(\mathbf{r}'_{pq})R_l^s(\mathbf{r}_q).$$



Translations of elementary solutions of the 3D Laplace equation (2)

$$S_n^m(\mathbf{r}_p) = \sum_{l=0}^{p} \sum_{s=-l}^{l} (S|R)_{ln}^{sm}(\mathbf{r}'_{pq}) R_l^s(\mathbf{r}_q), \qquad |\mathbf{r}_q| < |\mathbf{r}'_{pq}|, \quad p \neq q.$$

$$S_n^m(\mathbf{r}_p) = \sum_{l=0}^{p} \sum_{s=-l}^{l} (S|S)_{ln}^{sm}(\mathbf{r}'_{pq}) S_l^s(\mathbf{r}_q), \qquad |\mathbf{r}_q| > |\mathbf{r}'_{pq}|,$$

$$R_n^m(\mathbf{r}_p) = \sum_{l=0}^{p} \sum_{s=-l}^{l} (R|R)_{ln}^{sm}(\mathbf{r}'_{pq}) R_l^s(\mathbf{r}_q).$$

Number of translation coefficients $(S|R)_{ln}^{sm}, (S|S)_{ln}^{sm}(\mathbf{r}'_{pq})$, or $(R|R)_{ln}^{sm}(\mathbf{r}'_{pq})$:

$$P = 1 + 3 + ... + (2p + 1) = (p + 1)^2.$$

Differentiation of Multipoles

$$D_x = \frac{\partial}{\partial x}, \quad D_y = \frac{\partial}{\partial y}, \quad D_z = \frac{\partial}{\partial z}, \quad \text{or} \quad D_{\mathbf{t}} = \mathbf{t} \bullet \nabla,$$

$$G_n(\mathbf{r}) = (-1)^n D_{\mathbf{t}_1} D_{\mathbf{t}_2} \ldots D_{\mathbf{t}_n} \frac{1}{|\mathbf{r}|}, \quad |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \neq 0$$

$$G_{n+1}(\mathbf{r}) = (-1)^n D_{\mathbf{t}_1} D_{\mathbf{t}_2} \ldots D_{\mathbf{t}_n} D_{\mathbf{t}_{n+1}} \frac{1}{|\mathbf{r}|} = D_{\mathbf{t}_{n+1}} G_n(\mathbf{r}).$$

$$D_{\mathbf{t}_{n+1}} = \mathbf{t}_{n+1} \bullet \nabla = t_{n+1}^{(x)} D_x + t_{n+1}^{(y)} D_y + t_{n+1}^{(z)} D_z.$$

We introduce also operators

$$D_{x+iy} = D_x + iD_y, \quad D_{x-iy} = D_x - iD_y,$$

so

$$D_{\mathbf{t}_{n+1}} = \frac{1}{2} t_{n+1}^{(x)} [D_{x+iy} + D_{x-iy}] + \frac{1}{2i} t_{n+1}^{(y)} [D_{x+iy} - D_{x-iy}] + t_{n+1}^{(z)} D_z.$$

Operator D_z

$$\mathbf{r} = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

$$D_z = \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}, \quad \mu = \cos\theta.$$

We will use the following recursions for the Associated Legendre Functions

$$\mu P_n^m = \frac{1}{2n+1}[(n+m)P_{n-1}^m + (n-m+1)P_{n+1}^m],$$
$$(1-\mu^2)\frac{d}{d\mu}P_n^m = \frac{1}{2n+1}[(n+1)(n+m)P_{n-1}^m - n(n-m+1)P_{n+1}^m].$$

So

$$n\mu P_n^m + (1-\mu^2)\frac{d}{d\mu}P_n^m = (n+m)P_{n-1}^m,$$
$$-(n+1)\mu P_n^m + (1-\mu^2)\frac{d}{d\mu}P_n^m = -(n-m+1)P_{n+1}^m.$$

Operator D$_z$ (2)

$$D_z R_n^m(\mathbf{r}) = D_z[r^n Y_n^m(\theta,\varphi)] = \left(\mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\right)[r^n Y_n^m(\theta,\varphi)]$$

$$= r^{n-1}\left[n\mu Y_n^m + (1-\mu^2)\frac{\partial}{\partial \mu} Y_n^m\right]$$

$$= r^{n-1} e^{im\varphi}(-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\left[n\mu P_n^{|m|} + (1-\mu^2)\frac{d}{d\mu}P_n^{|m|}\right]$$

$$= r^{n-1} e^{im\varphi}(-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}(n+|m|)P_{n-1}^{|m|} - (n-m+1)P_{n-1}^m$$

$$= r^{n-1}\left[(-1)^m \sqrt{\frac{2n-1}{4\pi}\frac{(n-1-|m|)!}{(n-1+|m|)!}}\, e^{im\varphi} P_{n-1}^{|m|}\right]\sqrt{\frac{2n+1}{2n-1}\frac{(n-|m|)}{(n+|m|)}}(n+|m|)$$

$$= \sqrt{\frac{2n+1}{2n-1}(n^2-m^2)}\, r^{n-1} Y_{n-1}^m(\theta,\varphi) = \sqrt{\frac{2n+1}{2n-1}(n^2-m^2)}\, R_{n-1}^m(\mathbf{r}).$$

Operator D_z (3)

$$D_z S_n^m(\mathbf{r}) = D_z[r^{-n-1} Y_n^m(\theta,\varphi)] = \left(\mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\right)[r^{-n-1} Y_n^m(\theta,\varphi)]$$

$$= r^{-n-2}\left[-(n+1)\mu Y_n^m + (1-\mu^2)\frac{\partial}{\partial \mu} Y_n^m\right]$$

$$= r^{-n-2} e^{im\varphi}(-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\left[-(n+1)\mu P_n^{|m|} + (1-\mu^2)\frac{d}{d\mu} P_n^{|m|}\right]$$

$$= -r^{-n-2} e^{im\varphi}(-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}(n-|m|+1) P_{n+1}^{|m|}$$

$$= -r^{-n-2}\left[(-1)^m \sqrt{\frac{2n+3}{4\pi}\frac{(n+1-|m|)!}{(n+1+|m|)!}} e^{im\varphi} P_{n+1}^{|m|}\right]\sqrt{\frac{2n+1}{2n+3}\frac{(n+1+|m|)}{(n+1-|m|)}}(n+1-|m|)$$

$$= -\sqrt{\frac{2n+1}{2n+3}\left[(n+1)^2 - m^2\right]}\, r^{-n-2} Y_{n+1}^m(\theta,\varphi) = -\sqrt{\frac{2n+1}{2n+3}\left[(n+1)^2 - m^2\right]}\, S_{n+1}^m(\mathbf{r}).$$

Operator D_z (4)

Summary:

We proved that

$$D_z R_n^m(\mathbf{r}) = a_n^m R_{n-1}^m(\mathbf{r}), \quad D_z S_n^m(\mathbf{r}) = -a_{n+1}^m S_{n+1}^m(\mathbf{r}),$$

$$a_n^m = \sqrt{\frac{2n+1}{2n-1}(n^2 - m^2)},$$

where

$$Y_n^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\mu) e^{im\varphi}, \quad \mu = \cos\theta.$$

$$R_n^m(\mathbf{r}) = r^n Y_n^m(\theta,\varphi),$$

$$S_n^m(\mathbf{r}) = r^{-n-1} Y_n^m(\theta,\varphi).$$

Operators D_{x+iy} and D_{x-iy}

Summary:

$$D_{x+iy}R_n^m(\mathbf{r}) = b_n^m R_{n-1}^{m+1}(\mathbf{r}), \quad D_{x+iy}S_n^m(\mathbf{r}) = c_{n+1}^{-m-1} S_{n+1}^{m+1}(\mathbf{r}),$$

$$D_{x-iy}R_n^m(\mathbf{r}) = b_n^{-m} R_{n-1}^{m-1}(\mathbf{r}), \quad D_{x-iy}S_n^m(\mathbf{r}) = c_{n+1}^{m-1} S_{n+1}^{m-1}(\mathbf{r}),$$

$$b_n^m = \begin{cases} -\sqrt{\dfrac{(2n+1)(n-m-1)(n-m)}{(2n-1)}}, & 0 \leqslant m \leqslant n, \\ \sqrt{\dfrac{(2n+1)(n-m-1)(n-m)}{(2n-1)}}, & -n \leqslant m < 0 \\ 0, & |m| > n, \end{cases}$$

$$c_n^m = \begin{cases} \sqrt{\dfrac{(2n-1)(n-m-1)(n-m)}{(2n+1)}}, & 0 \leqslant m \leqslant n, \\ -\sqrt{\dfrac{(2n-1)(n-m-1)(n-m)}{(2n+1)}}, & -n \leqslant m < 0 \\ 0, & |m| > n, \end{cases}$$

Recursive Computation of Translation Coefficients

$$R_n^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(R|R)_{ln}^{sm}(\mathbf{r}'_{pq})R_l^s(\mathbf{r}_q).$$

$$D_z R_n^m(\mathbf{r}_p) = D_z R_n^m(\mathbf{r}_q + \mathbf{r}'_{pq}) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(R|R)_{ln}^{sm}(\mathbf{r}'_{pq})D_z R_l^s(\mathbf{r}_q),$$

Similarly,

$$D_z S_n^m(\mathbf{r}_p) = D_z S_n^m(\mathbf{r}_q + \mathbf{r}'_{pq}) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(S|R)_{ln}^{sm}(\mathbf{r}'_{pq})D_z R_l^s(\mathbf{r}_q), \quad |\mathbf{r}'_{pq}| > |\mathbf{r}_q|,$$

$$D_z S_n^m(\mathbf{r}_p) = D_z S_n^m(\mathbf{r}_q + \mathbf{r}'_{pq}) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(S|S)_{ln}^{sm}(\mathbf{r}'_{pq})D_z S_l^s(\mathbf{r}_q), \quad |\mathbf{r}'_{pq}| < |\mathbf{r}_q|.$$

Recurrence for R|R based on D_z

On one hand:

$$D_z R_n^m(\mathbf{r}_p) = a_n^m R_{n-1}^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l} a_n^m (R|R)_{l,n-1}^{sm}(\mathbf{r}'_{pq}) R_l^s(\mathbf{r}_q).$$

On the other hand

$$D_z R_n^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(R|R)_{ln}^{sm}(\mathbf{r}'_{pq}) D_z R_l^s(\mathbf{r}_q) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(R|R)_{ln}^{sm}(\mathbf{r}'_{pq}) a_l^s R_{l-1}^s(\mathbf{r}_p)$$

$$= \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(R|R)_{l+1,n}^{sm}(\mathbf{r}'_{pq}) a_{l+1}^s R_l^s(\mathbf{r}_p).$$

So:

$$a_{l+1}^s (R|R)_{l+1,n}^{sm}(\mathbf{r}'_{pq}) = a_n^m (R|R)_{l,n-1}^{sm}(\mathbf{r}'_{pq}).$$

Recurrence for S|R based on D_z

On one hand:

$$D_z S_n^m(\mathbf{r}_p) = -a_{n+1}^m S_{n+1}^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(-a_{n+1}^m)(S|R)_{l,n+1}^{sm}(\mathbf{r}'_{pq})R_l^s(\mathbf{r}_q).$$

On the other hand

$$D_z S_n^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(S|R)_{ln}^{sm}(\mathbf{r}'_{pq})D_z R_l^s(\mathbf{r}_q) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(S|R)_{ln}^{sm}(\mathbf{r}'_{pq})a_l^s R_{l-1}^s(\mathbf{r}_p)$$

$$= \sum_{l=0}^{\infty}\sum_{s=-l}^{l}(S|R)_{l+1,n}^{sm}(\mathbf{r}'_{pq})a_{l+1}^s R_l^s(\mathbf{r}_p).$$

So:

$$a_{l+1}^s(S|R)_{l+1,n}^{sm}(\mathbf{r}'_{pq}) = -a_{n+1}^m(S|R)_{l,n+1}^{sm}(\mathbf{r}'_{pq}).$$

Recurrence for S|S based on D_z

On one hand:

$$D_z S_n^m(\mathbf{r}_p) = -a_{n+1}^m S_{n+1}^m(\mathbf{r}_p) = -\sum_{l=0}^{\infty}\sum_{s=-l}^{l} a_{n+1}^m (S|S)_{l,n+1}^{sm}(\mathbf{r}'_{pq}) S_l^s(\mathbf{r}_q).$$

On the other hand

$$D_z S_n^m(\mathbf{r}_p) = \sum_{l=0}^{\infty}\sum_{s=-l}^{l} (S|S)_{ln}^{sm}(\mathbf{r}'_{pq}) D_z S_l^s(\mathbf{r}_q) = -\sum_{l=0}^{\infty}\sum_{s=-l}^{l} (S|R)_{ln}^{sm}(\mathbf{r}'_{pq}) a_{l+1}^s S_{l+1}^s(\mathbf{r}_p)$$

$$= -\sum_{l=0}^{\infty}\sum_{s=-l}^{l} (S|S)_{l-1,n}^{sm}(\mathbf{r}'_{pq}) a_l^s S_l^s(\mathbf{r}_p).$$

So:

$$a_{n+1}^m (S|S)_{l,n+1}^{sm}(\mathbf{r}'_{pq}) = a_l^s (S|S)_{l-1,n}^{sm}(\mathbf{r}'_{pq}).$$

More General Form of Recursions (valid for 3D Laplace and Helmholtz Equations)

$$\nabla^2 \Phi + k^2 \Phi = 0,$$

$$a_{n-1}^{m} (E|F)_{l,n-1}^{sm}(\mathbf{r}'_{pq}) - a_{n}^{m} (E|F)_{l,n+1}^{sm}(\mathbf{r}'_{pq}) =$$
$$a_{l}^{s} (E|F)_{l+1,n}^{sm}(\mathbf{r}'_{pq}) - a_{l-1}^{s} (E|F)_{l-1,n}^{sm}(\mathbf{r}'_{pq}), \quad E,F = S,R,$$

$$l,n = 0,1,\ldots \qquad s = -l,\ldots,l, \qquad m = -n,\ldots,n.$$

with

$$a_n^m = \begin{cases} \sqrt{\frac{(n+1+|m|)(n+1-|m|)}{(2n+1)(2n+3)}}, & n \geqslant |m|, \\ 0, & |m| > n. \end{cases}$$

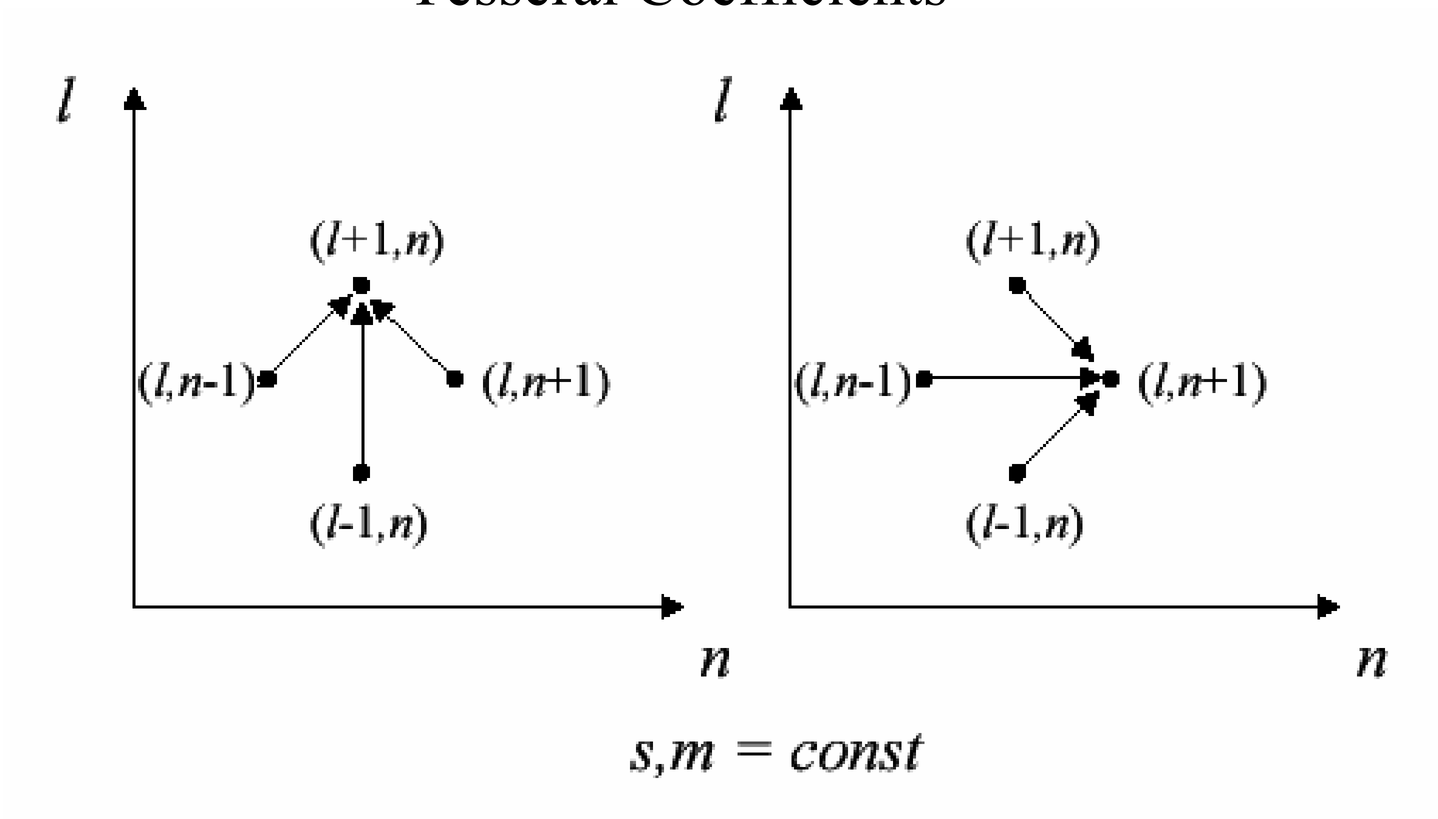
Other Recursions (valid for 3D Laplace and Helmholtz Equations) Can Be Found In Our Technical Report

Gumerov, N.A. & Duraiswami, R.,
``Fast, Exact, and Stable Computation of Multipole Translation and Rotation Coefficients for the 3-D Helmholtz Equation,"

University of Maryland Institute for Advanced Computer Studies Technical Report UMIACS-TR-\# 2001-44, Also CS-TR-\# 4264. (Available at http://www.cs.umd.edu/Library/TRs/CS-TR-4264/CS-TR-4264.pdf)

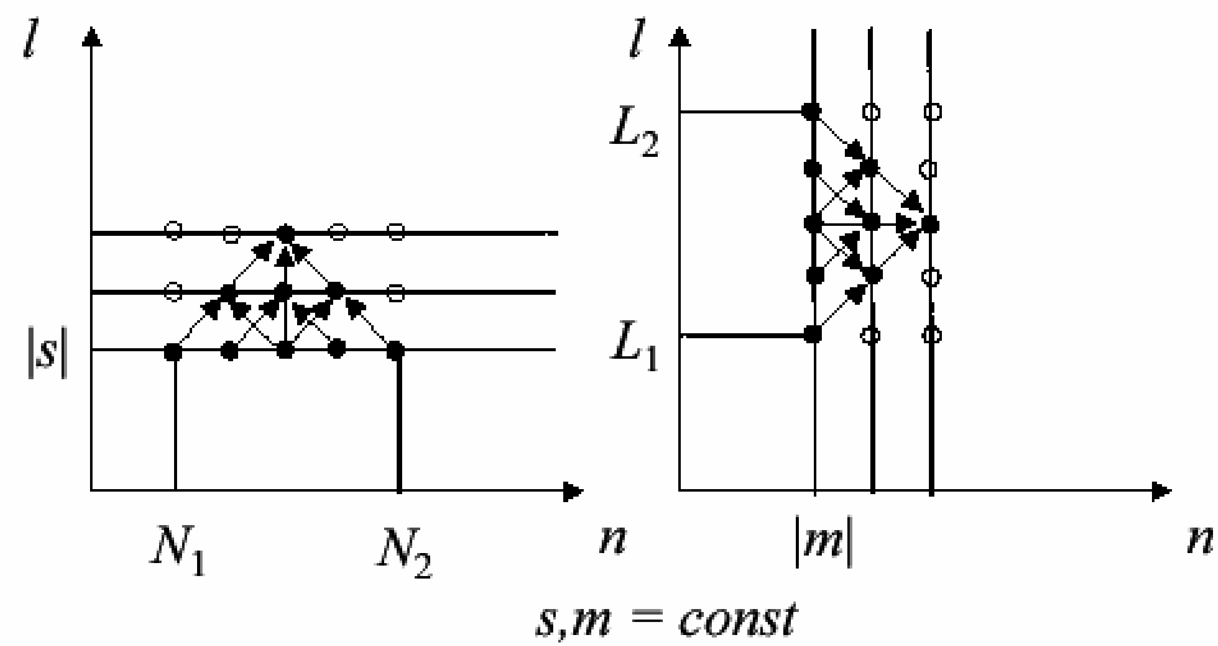
Recursive Computation of Translation Matrix, Example for (S|R) and 3D Helmholtz

Tesseral Coefficients



Recursive Computation of Translation Matrix, Example for (S|R) and 3D Helmholtz (2)

Tesseral Coefficients



Recursive Computation of Translation Matrix, Example for (S|R) and 3D Helmholtz (3)

Sectorial Coefficients

$$b_{m+1}^{-m-1}(S|R)_{l,m+1}^{s,m+1} = b_l^{-s}(S|R)_{l-1,m}^{s-1,m} - b_{l+1}^{s-1}(S|R)_{l+1,m}^{s-1,m},$$

$$b_{m+1}^{-m-1}(S|R)_{l,m+1}^{s,-m-1} = b_l^{s}(S|R)_{l-1,m}^{s+1,-m} - b_{l+1}^{-s-1}(S|R)_{l+1,m}^{s+1,-m},$$

$$l = 0, 1, \ldots \quad s = -l, \ldots, l, \quad m = 0, 1, 2, \ldots.$$

$$b_n^m = \begin{cases} \sqrt{\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}}, & 0 \leqslant m \leqslant n, \\ -\sqrt{\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}}, & -n \leqslant m < 0, \\ 0, & |m| > n, \end{cases}$$

and the recurrence process starts with

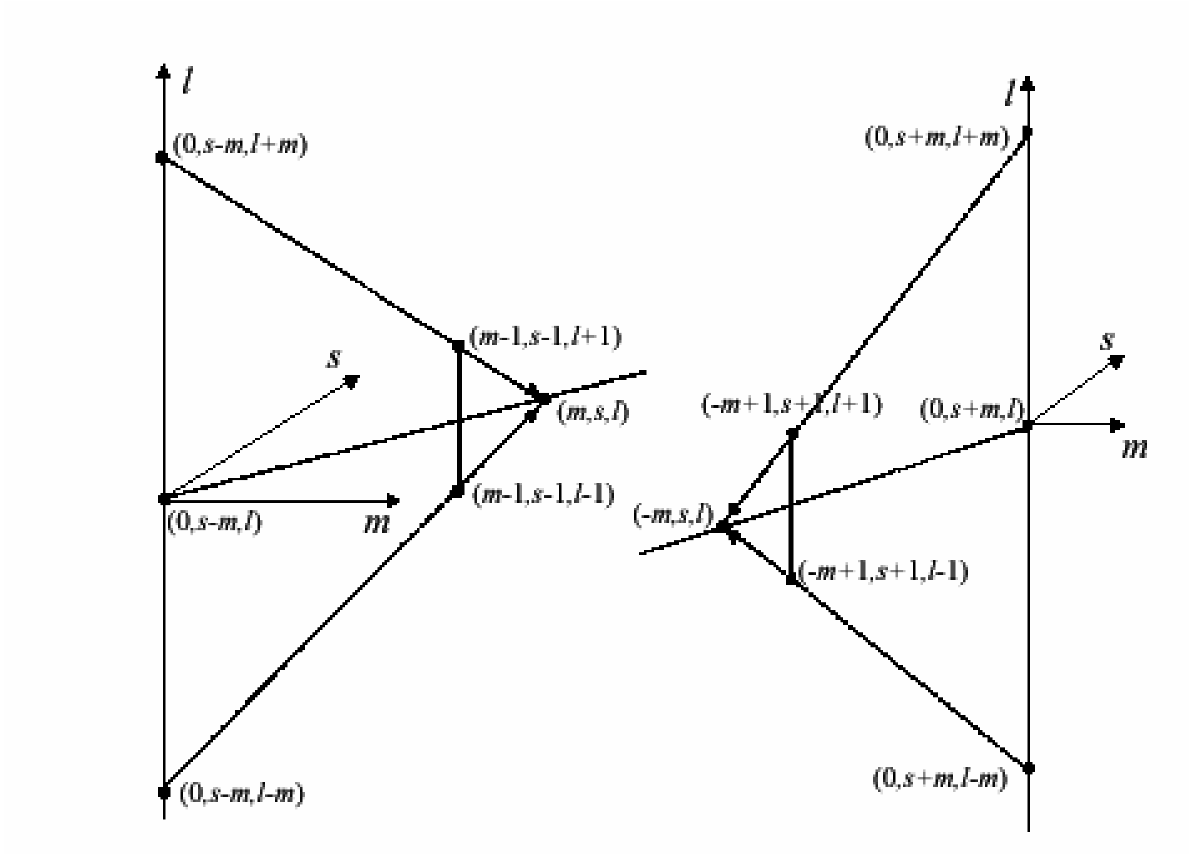
$$(S|R)_{l0}^{s0}(\mathbf{r}'_{pq}) = \sqrt{4\pi}\,(-1)^l S_l^{-s}(\mathbf{r}'_{pq}), \quad (S|R)_{0n}^{0m}(\mathbf{r}'_{pq}) = \sqrt{4\pi}\,S_n^m(\mathbf{r}'_{pq}).$$

Symmetry relation

$$(E|F)_{|m|l}^{-m,-s} = (-1)^{l+m}(E|F)_{l|m|}^{sm}, \quad l = 0, 1, 2, \ldots, \quad s = -l, \ldots, l, \quad m = -n, \ldots, n.$$

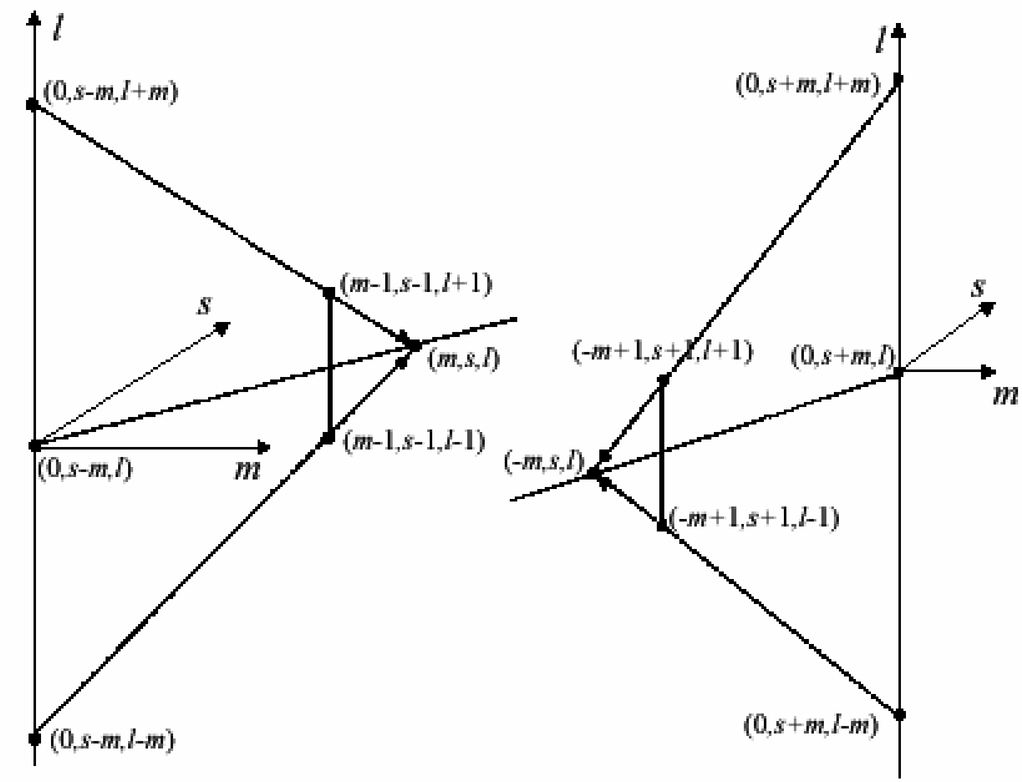
Recursive Computation of Translation Matrix, Example for (S|R) and 3D Helmholtz (4)

Sectorial Coefficients



Recursive Computation of Translation Matrix, Example for (S|R) and 3D Helmholtz (5)

Sectorial Coefficients



Coaxial Translation

Particular case when translation vector is aligned with the z –axis:

$$\mathbf{r}'_{pq} = r'_{pq}\mathbf{i}_z$$

$$(E|F)^{m}_{ln}(r'_{pq}) = (E|F)^{mm}_{ln}(\mathbf{r}'_{pq}), \quad l,n = 0,1,..., \quad m = -n,...,n, \quad (E|F) = (S|R),(S|S),(R|R)$$

$$S^{m}_{n}(\mathbf{r}_p) = \sum_{l=|m|}^{\infty}(S|R)^{m}_{ln}(r'_{pq})R^{s}_{l}(\mathbf{r}_q), \qquad |\mathbf{r}_q| < |\mathbf{r}'_{pq}|,$$

$$S^{m}_{n}(\mathbf{r}_p) = \sum_{l=|m|}^{\infty}(S|S)^{m}_{ln}(r'_{pq})S^{s}_{l}(\mathbf{r}_q), \qquad |\mathbf{r}_q| > |\mathbf{r}'_{pq}|,$$

$$R^{m}_{n}(\mathbf{r}_p) = \sum_{l=|m|}^{\infty}(R|R)^{m}_{ln}(r'_{pq})R^{s}_{l}(\mathbf{r}_q).$$

Rotations of coordinates

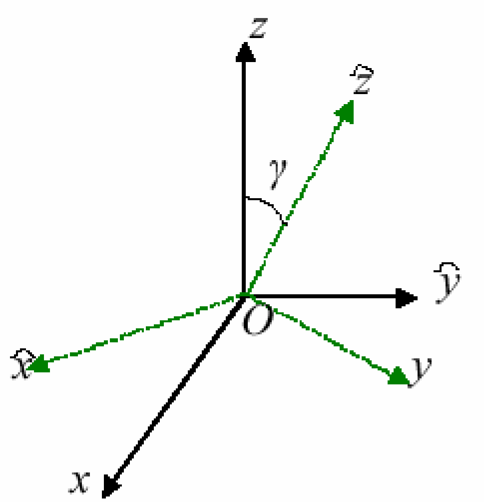
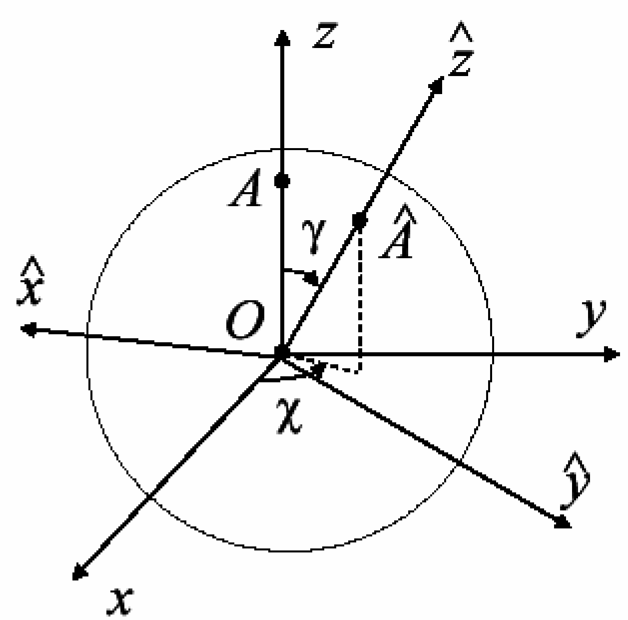
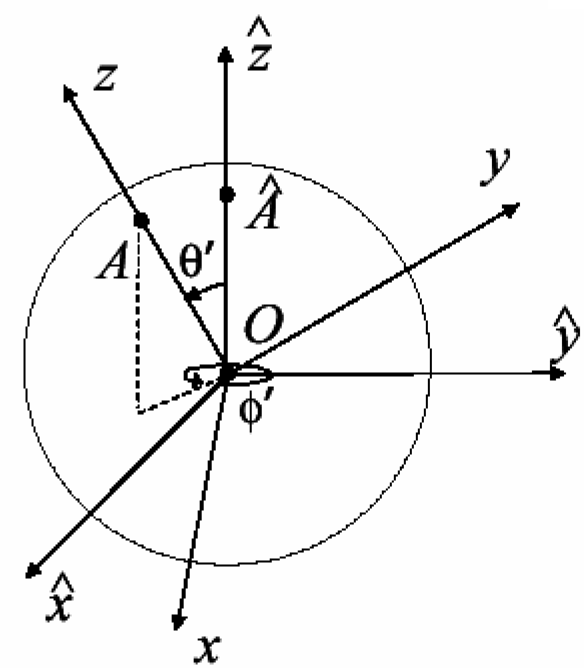


FIG. 6.1. *Rotation*

$$Q = \begin{bmatrix} \mathbf{i}_{\hat{x}} \cdot \mathbf{i}_x & \mathbf{i}_{\hat{x}} \cdot \mathbf{i}_y & \mathbf{i}_{\hat{x}} \cdot \mathbf{i}_z \\ \mathbf{i}_{\hat{y}} \cdot \mathbf{i}_x & \mathbf{i}_{\hat{y}} \cdot \mathbf{i}_y & \mathbf{i}_{\hat{y}} \cdot \mathbf{i}_z \\ \mathbf{i}_{\hat{z}} \cdot \mathbf{i}_x & \mathbf{i}_{\hat{z}} \cdot \mathbf{i}_y & \mathbf{i}_{\hat{z}} \cdot \mathbf{i}_z \end{bmatrix}$$





Rotations of elementary solutions of the 3D Laplace equation (also 3D Helmholtz)

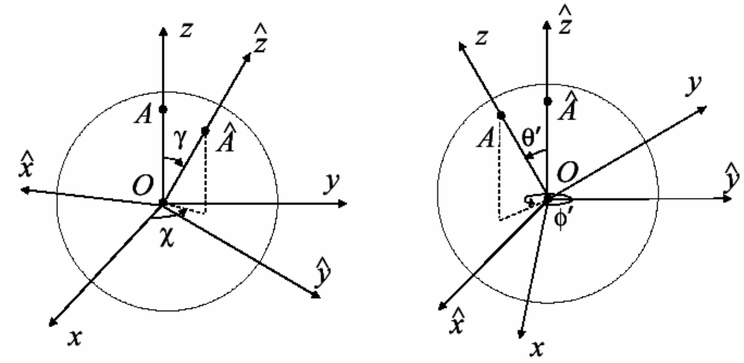
Rotations

$$Y_n^m(\theta,\varphi) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) Y_n^{\nu}\left(\hat{\theta},\hat{\varphi}\right),$$

$$S_n^m(\mathbf{r}_p) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) S_n^{\nu}(\hat{\mathbf{r}}_p), \qquad |\hat{\mathbf{r}}_p| = |\mathbf{r}_p|,$$

$$R_n^m(\mathbf{r}_p) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) R_n^{\nu}(\hat{\mathbf{r}}_p), \qquad |\hat{\mathbf{r}}_p| = |\mathbf{r}_p|,$$

Recursive Computation of Rotation Coefficients



$$Y_n^m(\theta,\varphi) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) Y_n^\nu\left(\hat{\theta},\hat{\varphi}\right),$$

$$T_n^{\nu m}(\theta',\varphi',\chi) = e^{im\chi} e^{-i\nu\varphi'} H_n^{\nu m}(\theta').$$

Recurrence relation

$$H_{n-1}^{\nu,m+1} = \frac{1}{b_n^m}\left\{\frac{1}{2}\left[b_n^{-\nu-1}(1-\cos\theta')H_n^{\nu+1,m} - b_n^{\nu-1}(1+\cos\theta')H_n^{\nu-1,m}\right] - a_{n-1}^\nu \sin\theta' H_n^{\nu m}\right\},$$

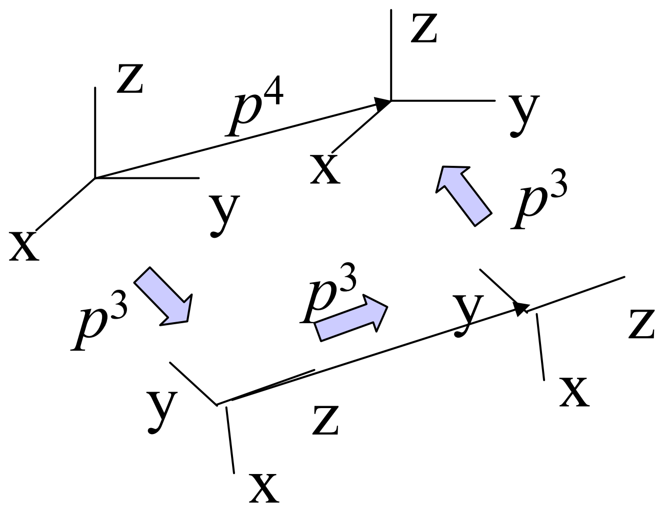
$$n = 2,3,\ldots,\quad \nu = -n+1,\ldots,n-1,\quad m = 0,\ldots,n-2.$$

Initial Values

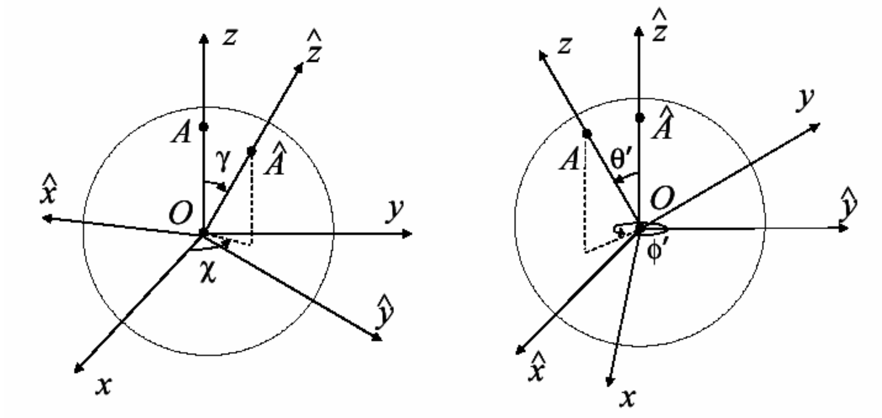
$$P_n(\cos\theta) = \frac{4\pi}{2n+1}\sum_{\nu=-n}^{n} Y_n^{-\nu}(\theta',\varphi') Y_n^\nu\left(\hat{\theta},\hat{\varphi}\right),$$

$$H_n^{\nu 0}(\theta') = (-1)^\nu \sqrt{\frac{(n-|\nu|)!}{(n+|\nu|)!}} P_n^{|\nu|}(\cos\theta'),\quad n = 0,1,\ldots,\quad \nu = -n,\ldots,n.$$

Rotation-Coaxial Translation Decomposition



Rotation



Coaxial Translation

$$S_n^m(\tilde{\mathbf r}+\mathbf i_z d) = \sum_{l=|m|}^{\infty}(S|R)_{ln}^m(d)R_l^m(\tilde{\mathbf r}), \qquad |\tilde{\mathbf r}| < d,$$

$$S_n^m(\tilde{\mathbf r}+\mathbf i_z d) = \sum_{l=|m|}^{\infty}(S|S)_{ln}^m(d)S_l^m(\tilde{\mathbf r}), \qquad |\tilde{\mathbf r}| > d,$$

$$R_n^m(\tilde{\mathbf r}+\mathbf i_z d) = \sum_{l=|m|}^{\infty}(R|R)_{ln}^m(d)R_l^m(\tilde{\mathbf r}).$$

$$(E|F)_{ln}^m(d) = (E|F)_{ln}^{mm}(\mathbf d)|_{\theta_{pq}=0}, \quad E,F = S,R.$$

$$Y_n^m(\theta,\varphi) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q)Y_n^{\nu}(\hat\theta,\hat\varphi),$$

$$Q = \begin{bmatrix} \mathbf i_{\hat x}\cdot\mathbf i_x & \mathbf i_{\hat x}\cdot\mathbf i_y & \mathbf i_{\hat x}\cdot\mathbf i_z \\ \mathbf i_{\hat y}\cdot\mathbf i_x & \mathbf i_{\hat y}\cdot\mathbf i_y & \mathbf i_{\hat y}\cdot\mathbf i_z \\ \mathbf i_{\hat z}\cdot\mathbf i_x & \mathbf i_{\hat z}\cdot\mathbf i_y & \mathbf i_{\hat z}\cdot\mathbf i_z \end{bmatrix}.$$

Decomposition into Subspaces

$$\Phi^p(\mathbf{r}) = \sum_{n=0}^{p}\sum_{m=-n}^{n} A_n^m F_n^m(\mathbf{r}) = \sum_{m=-p}^{p}\sum_{n=|m|}^{p} A_n^m F_n^m(\mathbf{r}) = \mathbf{A}\cdot\mathbf{F}, \quad F = S, R.$$

$$\mathbf{A} = \mathbf{A}^0 \oplus \mathbf{A}^{\pm 1} \oplus \ldots = \sum_{m=-\infty}^{\infty} \oplus \mathbf{A}^m,$$

where

$$\mathbf{A}^m = \left(A_{|m|}^m, A_{|m|+1}^m, A_{|m|+2}^m, \ldots\right)^T, \quad m = 0, \pm 1, \pm 2, \ldots,$$

and as the direct sum of finite blocks $\mathbf{A}_n$ corresponding to degree m :

$$\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1 \oplus \ldots = \sum_{n=0}^{\infty} \oplus \mathbf{A}_n,$$

where

$$\mathbf{A}_n = (A_n^{-n}, \ldots, A_n^n)^T, \quad n = 0, 1, 2, \ldots.$$

So the coaxial translation operator has invariant subspaces at fixed order, *m*, while the rotation operator has invariant subspaces at fixed degree, *n*.

Coaxial Translation:

$$(\mathbf{S}|\mathbf{R}) = (\mathbf{S}|\mathbf{R})^{0} \oplus (\mathbf{S}|\mathbf{R})^{\pm 1} \oplus \ldots = \sum_{m=-\infty}^{\infty} \oplus (\mathbf{S}|\mathbf{R})^{m},$$

Rotation

$$(\mathbf{S}|\mathbf{R}) = (\mathbf{S}|\mathbf{R})_{0} \oplus (\mathbf{S}|\mathbf{R})_{1} \oplus \ldots = \sum_{n=0}^{\infty} \oplus (\mathbf{S}|\mathbf{R})_{n},$$