Lecture 20, CMSC 898R, AMSC 698R

Outline

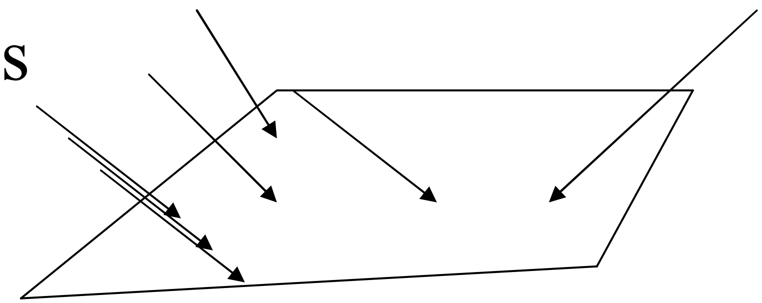
- Review
- Vector analysis (Divergence & Gradient of potential)
- 3-D Cartesian coordinates & Spherical coordinates
- Laplace’s equation
- Solution by separation of variables
- Elementary solutions in spherical coordinates
- Boundary conditions
- Green's function & Green's theorem
- Boundary element method
- Generation of multipoles from differentiation
- Translation theory ...
- Rotation and translation

Review of vector calculus

- Rate of change of a scalar function $R^d \to R$: $f(\boldsymbol{x})$

$$\nabla f = \sum_i \mathbf{e}_i \partial f / \partial x_i$$

- What about vector functions $R^d \to R^d$: $\boldsymbol{D}(\boldsymbol{x})$
- We can define several derivatives
  - Component wise gradient
    - Produces a higher order object
  - Divergence
  - Curl



- What is a vector function? E.g., velocity, or electric field
- A scalar quantity that can be defined is the "flux"
- - amount of stuff that crosses a surface

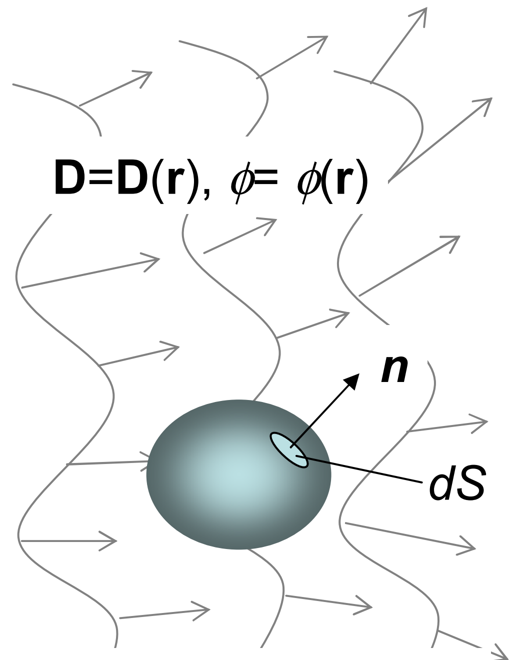
Definitions of *div*, *grad* and *curl*

- Also can define a vector quantity --- the cross product with the normal
- Taking a closed surface and shrinking it to zero we get

$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau \to 0} \frac{1}{\delta\tau} \oint_{\Delta S} \phi \mathbf{n} dS$$

$$\mathrm{div}\mathbf{D} \equiv \mathrm{Lim}_{\delta\tau \to 0} \frac{1}{\delta\tau} \oint_{\Delta S} \mathbf{D}.\mathbf{n} dS$$

$$\mathrm{curl}\mathbf{D} \equiv -\mathrm{Lim}_{\delta\tau \to 0} \frac{1}{\delta\tau} \oint_{\Delta S} \mathbf{D} \times \mathbf{n} dS$$



Elemental volume $\delta\tau$ with surface ΔS

Differential Viewpoint

- Divergence of vector at a point P is a scalar which measures the "spreading" of the vector field

$$\nabla \cdot \vec{\mathbf{F}} \quad = \quad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \vec{\mathbf{F}} \quad = \quad \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

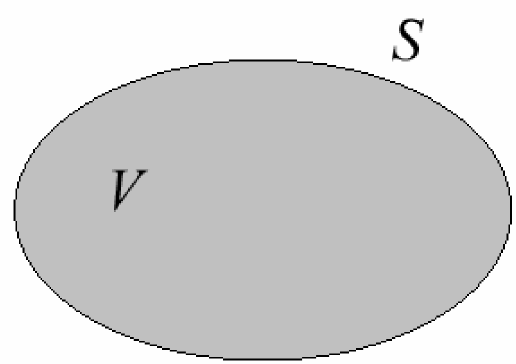
- Curl of a vector field at a point measures the amount of swirling or spinning.

$$\nabla \times \vec{\mathbf{F}} \quad = \quad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \times \vec{\mathbf{F}} \quad = \quad \left[\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\ \right]$$

Gauss Divergence theorem

- The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_V \nabla \bullet \overline{A}\, dV = \oint_S \overline{A} \bullet d\overline{S}$$



Proof follows from the definition of divergence.

- In practice we can write
- $\int_S \mathbf{n}$ (anything) $= \int_V \nabla$ (anything)

Laplace's Equation

- If there are no sources of material then flux will be zero
- So we have div(**u**) = $\nabla \cdot \mathbf{u} = 0$
- If **u** arises as the gradient of a scalar function

$$\mathbf{u} = \nabla f$$

- Then div(**u**) = $\nabla \cdot \nabla f = \nabla^2 f = 0$
- To obtain f inside a domain we can solve Laplace's equation with boundary conditions

Cartesian Coordinates

$u^1 = x, \quad -\infty < x < +\infty,$

$u^2 = y, \quad -\infty < y < +\infty,$

$u^3 = z, \quad -\infty < z < +\infty.$

Surfaces of constant x, y, or z are mutually orthogonal planes (Fig. 1.01).

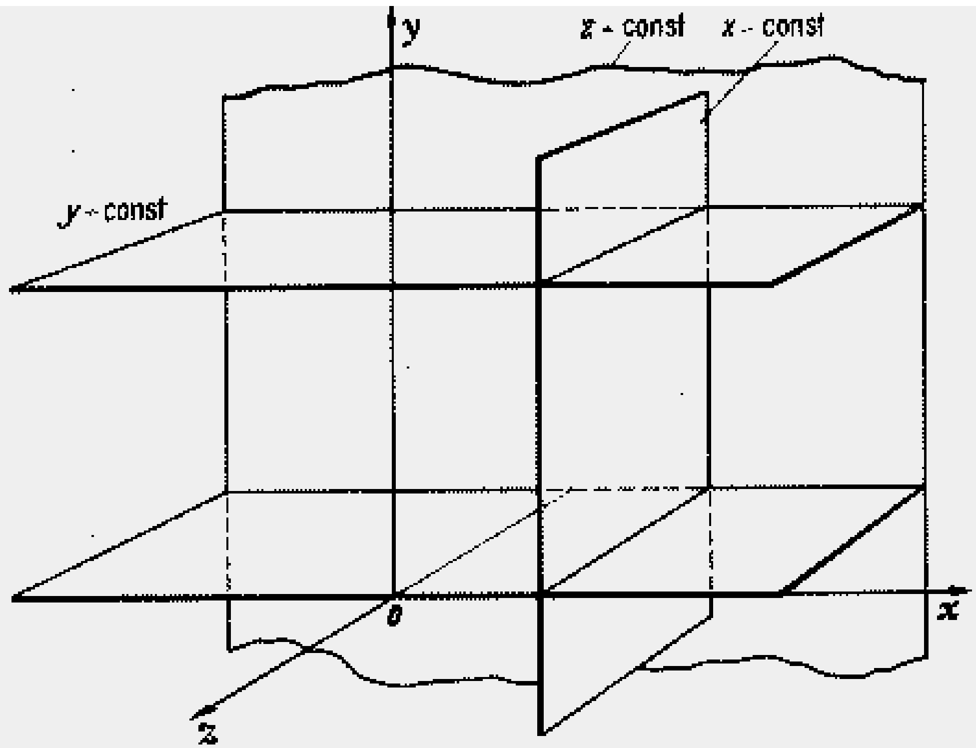


Fig. 1.01. Rectangular coordinates (x, y, z). Coordinate surfaces are the planes: $x = \text{const}$, $y = \text{const}$, $z = \text{const}$

- Nabla operator: $\nabla = \mathbf{e}_x \partial/\partial x + \mathbf{e}_y \partial/\partial y + \mathbf{e}_z \partial/\partial z$
- Laplacian: $\nabla \cdot \nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

Spherical coordinates

- $\nabla = \mathbf{e}_r \partial/\partial r \; + \mathbf{e}_{\boldsymbol{\theta}}/r \; \partial/\partial r + \mathbf{e}_\varphi/(r \sin \theta) \; \partial/\partial \varphi$
- $\nabla^2 = \nabla \cdot \nabla = \partial^2/\partial r^2 + 2/r \; \partial/\partial r + 1/r^2 \; \partial^2/\partial \theta^2 + (r \sin \theta)^{-2} \; \partial^2/\partial \varphi^2$

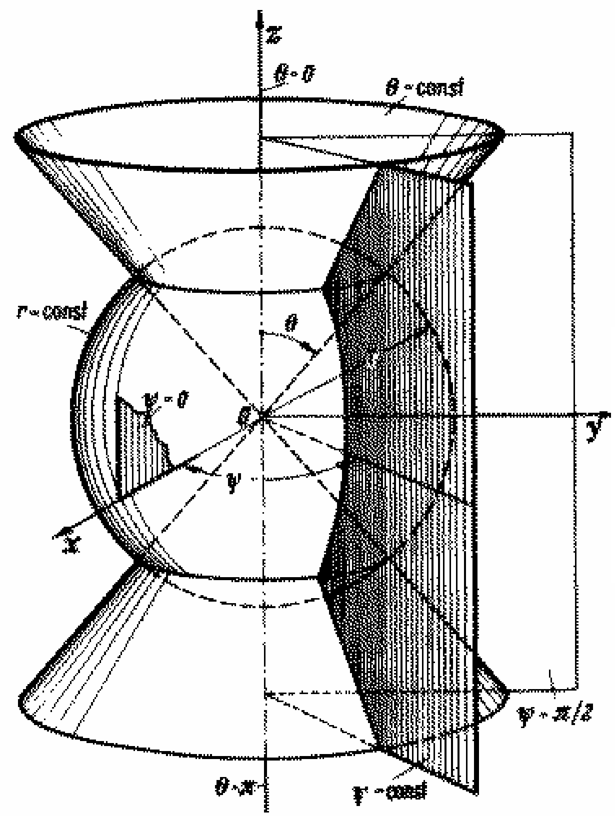


Fig. 1.05. Spherical coordinates (r, θ, ψ). Coordinate surfaces are spheres (r = const), circular cones (θ = const), and half planes (ψ = const)

$$u^1 = r, \qquad 0 \leqq r < \infty,$$
$$u^2 = \theta, \qquad 0 \leqq \theta \leqq \pi,$$
$$u^3 = \psi, \qquad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = r \sin\theta \cos\psi, \\ y = r \sin\theta \sin\psi, \\ z = r \cos\theta. \end{cases}$$

Coordinate surfaces are

),

(half planes, ψ = const).

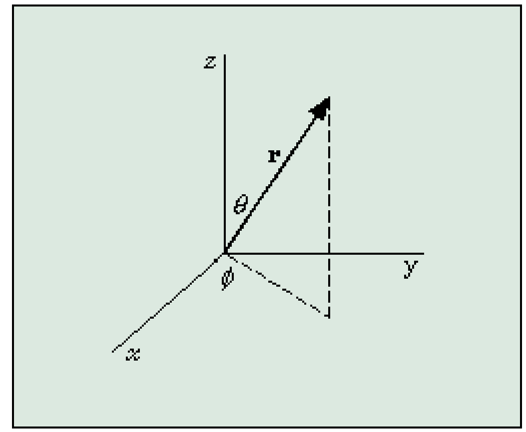
The Stäckel matrix may be written

$$[S] = \begin{bmatrix} 1 & -1/r^2 & 0 \\ 0 & 1 & -1/\sin^2\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = 1, \qquad M_{11} = 1,$$

Spherical coordinates

- $x = r \sin\theta \cos\varphi$
- $y = r \sin\theta \sin\varphi$
- $z = r \cos\theta$
- $r = (x^2+y^2+z^2)^{1/2}$
- $\varphi = \tan^{-1}(y/x)$
- $\theta = \tan^{-1}(x^2+y^2)^{1/2}/z$
- Gradient operator:



Differential Forms of the Gradient

$$grad\Phi = \nabla\Phi$$

$$\vec{i}\,\frac{\partial\Phi}{\partial x} + \vec{j}\,\frac{\partial\Phi}{\partial y} + \vec{k}\,\frac{\partial\Phi}{\partial z} \qquad \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\Phi$$

$$\vec{e}_r\,\frac{\partial\Phi}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \vec{e}_z\,\frac{\partial\Phi}{\partial z} \qquad = \left(\vec{e}_r\,\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \vec{e}_z\,\frac{\partial}{\partial z}\right)\Phi$$

$$\vec{e}_r\,\frac{\partial\Phi}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} \qquad \left(\vec{e}_r\,\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\Phi$$

Differential Forms of the Divergence

$$div\vec{A} = \nabla.\vec{A}$$

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{A}$$

$$\frac{1}{r}\frac{\partial rA_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \qquad = \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta} + \vec{e}_z\frac{\partial}{\partial z}\right).\vec{A}$$

$$\frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta \sin\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right).\vec{A}$$

Differential Forms of the Curl

$$curl\vec{A} \equiv -\mathrm{Lim}_{\delta\tau \to 0} \frac{1}{\delta\tau} \oint_{\Delta S} \vec{A} \times \vec{n}\, \mathrm{dS}$$

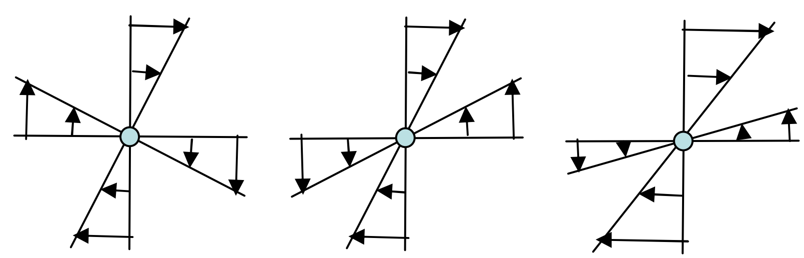
$$curl\vec{A} = \nabla \times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

Curl of the velocity vector $\nabla \times V$ **=**
twice the circumferentially averaged angular velocity of

-the flow around a point, or

-a fluid particle

=Vorticity Ω



Separation of Variables

- Usual way to solve Laplace's equation on simple geometries
- Assume a solution of the form *X(x)Y(y)Z(z)*
  - (Factorization just like FMM)
- Get three equations and Solve them
- Apply boundary conditions to fix coefficients/constants

If $\alpha_2 = p^2$ *and* $\alpha_3 = q^2$,

$$\frac{d^2X}{dx^2} - (p^2 + q^2)\,X = 0, \quad \{04\}^* \quad X = A\,e^{(p^2+q^2)^{\frac{1}{2}} x} + B\,e^{-(p^2+q^2)^{\frac{1}{2}} x}.$$

$$\frac{d^2Y}{dy^2} + p^2\,Y = 0, \quad \{04\} \quad Y = A\sin p\,y + B\cos p\,y.$$

$$\frac{d^2Z}{dz^2} + q^2 Z = 0, \quad \{04\} \quad Z = A\sin q\,z + B\cos q\,z.$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = -q^2$,

$$\frac{d^2X}{dx^2} - (p^2 - q^2)\,X = 0, \quad \{04\} \quad X = A\,e^{(p^2-q^2)^{\frac{1}{2}} x} + B\,e^{-(p^2-q^2)^{\frac{1}{2}} x}.$$

$$\frac{d^2Y}{dy^2} + p^2\,Y = 0, \quad \{04\} \quad Y = A\sin p\,y + B\cos p\,y.$$

$$\frac{d^2Z}{dz^2} - q^2 Z = 0, \quad \{04\} \quad Z = A\,e^{qz} + B\,e^{-qz}.$$

Spherical Coordinates

- Assume solution is of the form R(r)Θ(θ)Φ(φ)
- Solve

General case

$$\begin{cases} \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\alpha_2}{r^2}R = 0, \\ \frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left(\alpha_2 - \frac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \frac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q^2$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{p(p+1)}{r^2}R = 0, \quad \{22\} \quad R = A\,r^p + B\,r^{-(p+1)}.$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0, \quad \{222\} \quad \Theta = A\mathscr{P}_p^q(\cos\theta) + B\mathscr{Q}_p^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin q\psi + B\cos q\psi.$$

Fundamental Solution

- Response to δ function input
- $\nabla^2 G = \delta$
- $\int f(x') \, \delta \, (x-x') \, d^3 x = f(x)$

$$P(\mathbf{x}) = -\int_S \left[\frac{\partial P}{\partial n}(\mathbf{y}) G(\mathbf{x},\mathbf{y}) - P(\mathbf{y}) \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) \right] dS(\mathbf{y}), \qquad \mathbf{x} \notin \Omega,$$