

FMM CMSC 878R/AMSC 698R

Lecture 18

Outline

- Example problem
 - S-expansion error;
 - S|S-translation error;
 - S|R-translation error;
 - R|R-translation error.
- Error and Neighborhoods
- Optimization of MLFMM within error bounds

Example Problem

Problem:

Evaluate the MLFMM error for computation of function'

$$v(\mathbf{y}) = \sum_{k=0}^N u_k \Phi(\mathbf{y}, \mathbf{x}_k),$$

$$\Phi(\mathbf{y}, \mathbf{x}_k) = \frac{1}{\mathbf{y} - \mathbf{x}_k},$$

where \mathbf{y} and \mathbf{x}_k are points in a box of size D and space is subdivided by the binary tree to the maximum level L .

This example is also good to evaluate 2D problem, by treating x and y as complex numbers!

From Lecture 6...

$$|y - x_*| < |x_i - x_*| :$$

↓ R-expansion

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y - x_*),$$

$$a_m(x_i, x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, \dots,$$

$$R_m(y - x_*) = (y - x_*)^m, \quad m = 0, 1, \dots$$

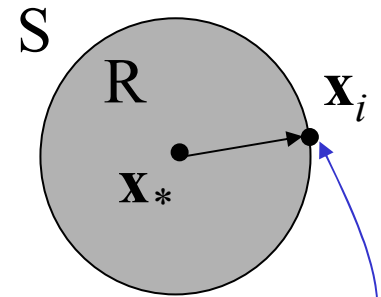
$$|y - x_*| > |x_i - x_*| :$$

↓ S-expansion

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*),$$

$$b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \dots,$$

$$S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \dots$$



Singular Point is located at the Boundary of regions for the R- and S-expansions!

From Lecture 8...

$$(|y - x_*| < |t|)$$

$$\begin{aligned} S_n(y - x_* + t) &= (t + y)^{-n-1} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} (y - x_*)^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} R_m(y - x_*) = \sum_{m=0}^{\infty} (S|R)_{mn}(t) R_m(y - x_*). \end{aligned}$$

So

$$(S|R)_{mn}(t) = \frac{1}{m!} \frac{d^m S_n(t)}{dt^m} = \frac{(-1)^m (m+n)!}{m! n! t^{n+m+1}}.$$

$$(S|R)(t) = \begin{pmatrix} t^{-1} & t^{-2} & t^{-3} & \dots \\ -t^{-2} & -2t^{-3} & -3t^{-4} & \dots \\ t^{-3} & 3t^{-4} & 6t^{-5} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

S|S-operator

$$S_n(y-x_{*1}) = \sum_{m=0}^{\infty} (S|S)_{mn}(t) S_m(y-x_{*2}), \quad t = x_{*2} - x_{*1}$$

$$\frac{1}{(1-\alpha)^{n+1}} = 1 + (n+1)\alpha + \frac{(n+1)(n+2)}{2!}\alpha^2 + \dots = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!}\alpha^m, \quad |\alpha| < 1.$$

$$\begin{aligned} S_n(y-x_{*1}) &= (y-x_{*1})^{-n-1} = (y-x_{*2} - (x_{*1} - x_{*2}))^{-n-1} \\ &= (y-x_{*2})^{-n-1} \left[1 - \frac{x_{*1} - x_{*2}}{y-x_{*2}} \right]^{-n-1} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)!}{m!n!} (x_{*2} - x_{*1})^m (y-x_{*2})^{-n-m-1} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)!}{m!n!} t^m S_{n+m}(y-x_{*2}) = \sum_{m=n}^{\infty} \frac{(-1)^{m-n} m!}{n!(m-n)!} t^{m-n} S_m(y-x_{*2}) \end{aligned}$$

$$(S|S)_{mn}(t) = \begin{cases} 0, & m < n \\ \frac{(-1)^{m-n} m!}{n!(m-n)!} t^{m-n}, & m \geq n. \end{cases}$$

S|S-operator (2)

$$S_n(y-x_{*1}) = \sum_{m=0}^{\infty} (S|S)_{mn}(t) S_m(y-x_{*2}), \quad t = x_{*2} - x_{*1}$$

$$(S|S)_{mn}(t) = \begin{cases} 0, & m < n \\ \frac{(-1)^{m-n} m!}{n!(m-n)!} t^{m-n}, & m \geq n. \end{cases}$$

$$(S|S)(t) = (S|S)_{mn}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -t & 1 & 0 & 0 & \dots \\ t^2 & -2t & 1 & 0 & \dots \\ -t^3 & 3t^2 & -3t & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

R|R-operator

$$R_n(\mathbf{y}-\mathbf{x}_{*1}) = \sum_{m=0}^{\infty} (R|R)_{mn}(\mathbf{t}) R_m(\mathbf{y}-\mathbf{x}_{*2}), \quad \mathbf{t} = \mathbf{x}_{*2} - \mathbf{x}_{*1}$$

$$\begin{aligned} R_n(\mathbf{y}-\mathbf{x}_{*1}) &= (\mathbf{y} - \mathbf{x}_{*1})^n = (\mathbf{y} - \mathbf{x}_{*2} - (\mathbf{x}_{*1} - \mathbf{x}_{*2}))^n \\ &= \sum_{m=0}^n \frac{(-1)^m n!}{m!(n-m)!} (\mathbf{x}_{*2} - \mathbf{x}_{*1})^{n-m} (\mathbf{y} - \mathbf{x}_{*2})^m \\ &= \sum_{m=0}^n \frac{(-1)^m n!}{m!(n-m)!} \mathbf{t}^{n-m} R_m(\mathbf{y}-\mathbf{x}_{*2}). \end{aligned}$$

$$(R|R)_{mn}(\mathbf{t}) = \begin{cases} 0, & m > n \\ \frac{(-1)^m n!}{m!(n-m)!} \mathbf{t}^{n-m}, & m \leq n. \end{cases}$$

R|R-operator(2)

$$R_n(y-x_{*1}) = \sum_{m=0}^n (R|R)_{mn}(t) R_m(y-x_{*2}), \quad t = x_{*2} - x_{*1}$$

$$(R|R)_{mn}(t) = \begin{cases} 0, & m > n \\ \frac{(-1)^m n!}{m!(n-m)!} t^{n-m}, & m \leq n. \end{cases}$$

$$(\mathbf{R}|\mathbf{R})(t) = (R|R)_{mn}(t) = \begin{pmatrix} 1 & t & t^2 & t^3 & \dots \\ 0 & -1 & -2t & -3t^2 & \dots \\ 0 & 0 & 1 & 3t & \dots \\ 0 & 0 & 0 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

S-Expansion Error

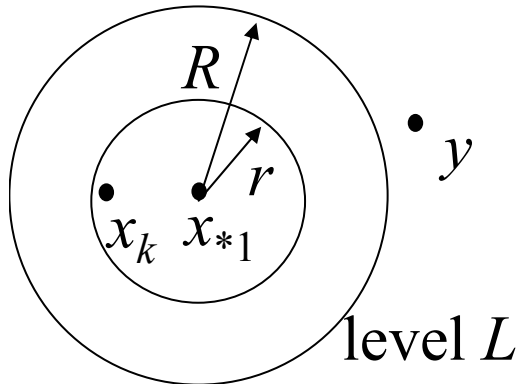
$$\Phi(y, x_k) = \frac{1}{y - x_k} = \sum_{m=0}^{\infty} b_m(x_k, x_{*1}) S_m(y - x_{*1}),$$

$$b_m(x_k, x_{*1}) = (x_k - x_{*1})^m, \quad m = 0, 1, \dots,$$

$$S_m(y - x_{*1}) = (y - x_{*1})^{-m-1}, \quad m = 0, 1, \dots$$

Assume

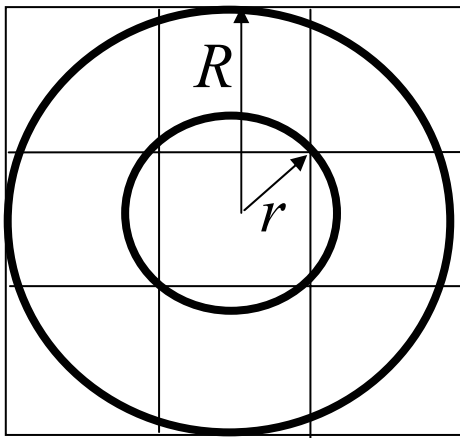
$$|x_k - x_{*1}| \leq r, \quad |y - x_{*1}| \geq R.$$



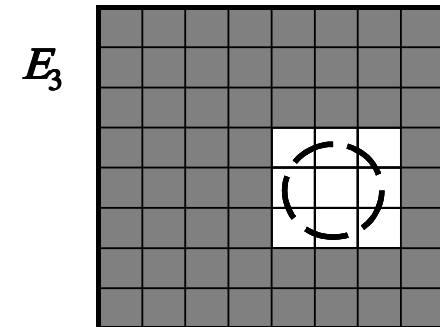
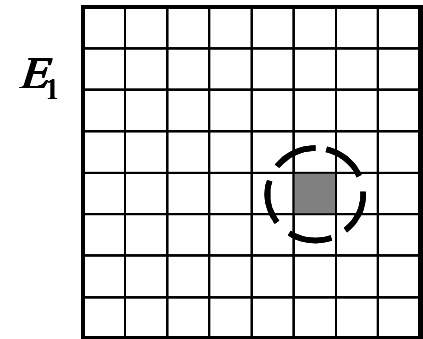
$$\begin{aligned} \text{ExpansionError}(p, R, r) &\leq \left| \sum_{m=0}^{\infty} b_m(x_k, x_{*1}) S_m(y - x_{*1}) - \sum_{m=0}^{p-1} b_m(x_k, x_{*1}) S_m(y - x_{*1}) \right| \\ &= \left| \sum_{m=p}^{\infty} b_m(x_k, x_{*1}) S_m(y - x_{*1}) \right| \leq \sum_{m=p}^{\infty} |b_m(x_k, x_{*1}) S_m(y - x_{*1})| \\ &= \frac{1}{|y - x_{*1}|} \sum_{m=p}^{\infty} \left(\frac{r}{R}\right)^m \leq \frac{1}{R} \left(\frac{r}{R}\right)^p \frac{1}{1 - r/R} = \left(\frac{r}{R}\right)^p \frac{1}{R - r}. \end{aligned}$$

S-Expansion Error

$$\text{ExpansionError}(p, R, r) \leq \left(\frac{r}{R}\right)^p \frac{1}{R-r}.$$



$$d = 1: r/R = 1/3,$$
$$d = 2: r/R = \sqrt{2}/3 < 1/2.$$



S|S-Translation Error

Translation from level $\alpha+1$ to α :

$$\hat{\mathbf{C}}^{(\alpha)} = (\mathbf{S}|\mathbf{S})(t)\hat{\mathbf{C}}^{(\alpha+1)}$$

$$\begin{pmatrix} \hat{C}_0^{(\alpha)} \\ \hat{C}_1^{(\alpha)} \\ \hat{C}_2^{(\alpha)} \\ \hat{C}_3^{(\alpha)} \\ \dots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -t & 1 & 0 & 0 & \dots \\ t^2 & -2t & 1 & 0 & \dots \\ -t^3 & 3t^2 & -3t & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \hat{C}_0^{(\alpha+1)} \\ \hat{C}_1^{(\alpha+1)} \\ \hat{C}_2^{(\alpha+1)} \\ \hat{C}_3^{(\alpha+1)} \\ \dots \end{pmatrix}$$

$$\begin{aligned} \hat{C}_0^{(\alpha)} &= \hat{C}_0^{(\alpha+1)}, \\ \hat{C}_1^{(\alpha)} &= -t\hat{C}_0^{(\alpha+1)} + \hat{C}_1^{(\alpha+1)}, \\ \hat{C}_2^{(\alpha)} &= t^2\hat{C}_0^{(\alpha+1)} - 2t\hat{C}_1^{(\alpha+1)} + \hat{C}_2^{(\alpha+1)}, \\ &\dots \end{aligned}$$

p first coefficients at level α can be exactly computed from p first coefficients at level $\alpha+1$.

This is exact translation of first p coefficients!

S|S-Translation Error(2)

Translation from level $\alpha+1$ to α :

$$\widehat{C}_m^{(\alpha)} = \sum_{n=0}^{p-1} (S|S)_{mn}(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)}) \widehat{C}_n^{(\alpha+1)} = \sum_{n=0}^{\infty} (S|S)_{mn}(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)}) \widehat{C}_n^{(\alpha+1)}$$

since $(S|S)_{mn}(t) = 0, \quad m < n \leq p.$

So:

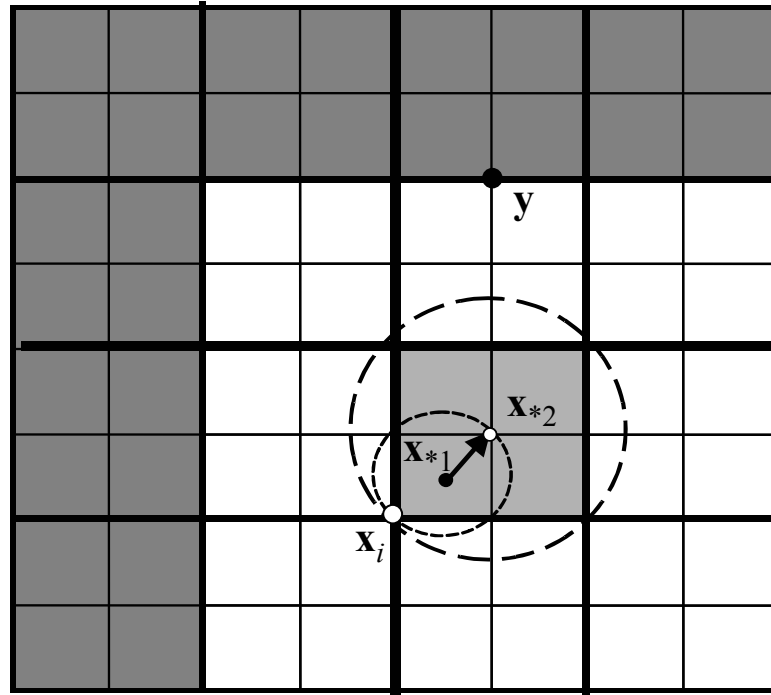
$$\begin{aligned} \left| \Phi(\mathbf{y}, \mathbf{x}_k) - \widehat{\Phi}_\alpha(\mathbf{y}, \mathbf{x}_k) \right| &\leq \left(\frac{2^{L-\alpha} r}{2^{L-\alpha} R} \right)^p \frac{1}{2^{L-\alpha} (R-r)} \\ &\leq \left(\frac{r}{R} \right)^p \frac{1}{(R-r)} = \text{ExpansionError}(p, R, r) \end{aligned}$$

This factor shows that we are on level α

For any level α !

S|S-Translation Error(3)

$$\begin{aligned} |\Phi(\mathbf{y}, \mathbf{x}_k) - \widehat{\Phi}_a(\mathbf{y}, \mathbf{x}_k)| &\leq \left(\frac{2^{L-a} r}{2^{L-a} R} \right)^p \frac{1}{2^{L-a} (R-r)} \\ &\leq \left(\frac{r}{R} \right)^p \frac{1}{(R-r)} = \text{ExpansionError}(p, R, r) \end{aligned}$$



In this example S|S-translations do not cause any additional error!

S|R-Translation Error (see homework 4)

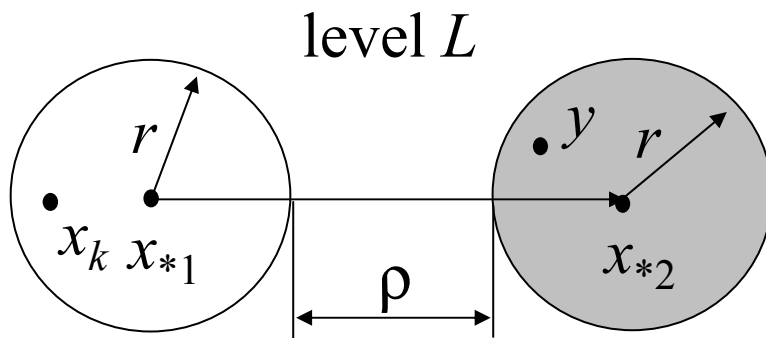
Note that in result of S|S-translations, first p coefficients are exact!

$$\begin{aligned} c_{mn} &= (S|R)_{mn}(t) b_n(x_k, x_{*1}) R_m(y - x_{*2}) \\ &= \frac{(-1)^m (m+n)!}{m! n! t^{m+n+1}} (x_k - x_{*1})^n (y - x_{*2})^m \end{aligned}$$

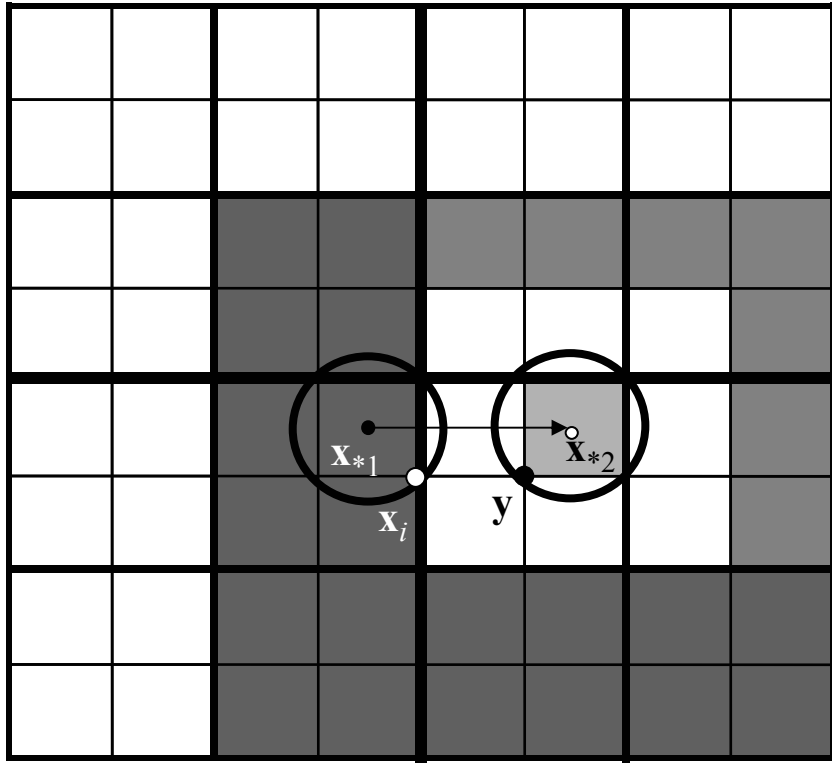
$$|x_k - x_{*1}| \leq 2^{L-l} r, \quad |y - x_{*2}| \leq 2^{L-l} r, \quad |x_{*2} - x_{*1}| \geq 2^{L-l} r + 2^{L-l} r + 2^{L-l} \rho$$

Translation with p -truncated operator $(S|R)_{mn}^{(p)}(t)$ yields

$$\Psi^{(l)}(y, x_k) = \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn}.$$



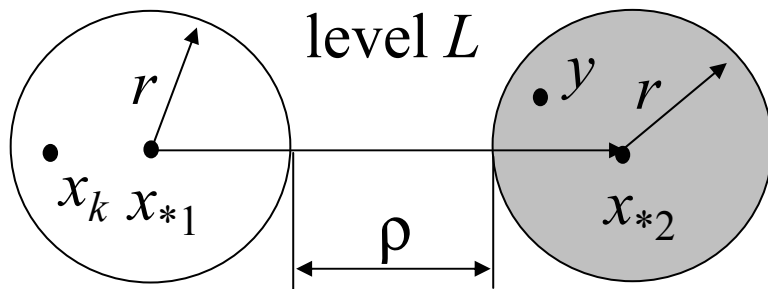
S|R-Translation Error (2)



$$d = 1: \rho / r = 2,$$

$$d = 2: \rho / r = 2(2 - \sqrt{2}) / \sqrt{2}$$

$$= 2(\sqrt{2} - 1) > 0.8$$



S|R-Translation Error (3)

$$\begin{aligned}
 |\Phi(y, x_k) - \Psi^{(l)}(y, x_k)| &= \left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} - \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn} \right| \\
 &= \left| \sum_{m=0}^{p-1} \sum_{n=0}^{\infty} c_{mn} + \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} c_{mn} - \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn} \right| \\
 &= \left| \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn} + \sum_{m=0}^{p-1} \sum_{n=p}^{\infty} c_{mn} + \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} c_{mn} - \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} c_{mn} \right| \\
 &= \left| \sum_{m=0}^{p-1} \sum_{n=p}^{\infty} c_{mn} + \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} c_{mn} \right| \leq \sum_{m=0}^{p-1} \sum_{n=p}^{\infty} |c_{mn}| + \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| \\
 &\leq \sum_{m=0}^{\infty} \sum_{n=p}^{\infty} |c_{mn}| + \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| = \sum_{n=p}^{\infty} \sum_{m=0}^{\infty} |c_{mn}| + \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| \\
 &= \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} |c_{nm}| + \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| = \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} (|c_{nm}| + |c_{mn}|)
 \end{aligned}$$

Long one!

continued
→

S|R-Translation Error (4)

$$\begin{aligned}
 &= \frac{1}{|t|} \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!} \left[\frac{|x_k - x_{*1}|^n |y - x_{*2}|^m + |x_k - x_{*1}|^m |y - x_{*2}|^n}{|t|^{m+n}} \right] \\
 &\leq \frac{1}{(2^{L-l}r + 2^{L-l}r + 2^{L-l}\rho)} \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!} \frac{(2^{L-l}r)^n (2^{L-l}r)^m + (2^{L-l}r)^m (2^{L-l}r)^n}{(2^{L-l}r + 2^{L-l}r + 2^{L-l}\rho)^{n+m}} \\
 &= \frac{1}{2^{L-l}r(2 + \frac{\rho}{r})} \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!} \frac{2r^{n+m}}{r^{n+m}(2 + \frac{\rho}{r})^{n+m}} \\
 &= \frac{2}{2^{L-l}r(2 + \frac{\rho}{r})} \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!} \frac{1}{(2 + \frac{\rho}{r})^{n+m}} \\
 &= \frac{2}{2^{L-l}r(2 + \frac{\rho}{r})} \sum_{m=p}^{\infty} \frac{1}{(2 + \frac{\rho}{r})^m} \sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!} \frac{1}{(2 + \frac{\rho}{r})^n} \\
 &= \frac{2}{2^{L-l}r(2 + \frac{\rho}{r})} \sum_{m=p}^{\infty} \frac{1}{(2 + \frac{\rho}{r})^m} \frac{1}{\left(1 - \frac{1}{2+\rho/r}\right)^{m+1}}
 \end{aligned}$$

It is really long!

we used this

continued

$$\frac{1}{(1-\alpha)^{n+1}} = 1 + (n+1)\alpha + \frac{(n+1)(n+2)}{2!}\alpha^2 + \dots = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} \alpha^m, \quad |\alpha| < 1.$$

S|R-Translation Error (5)

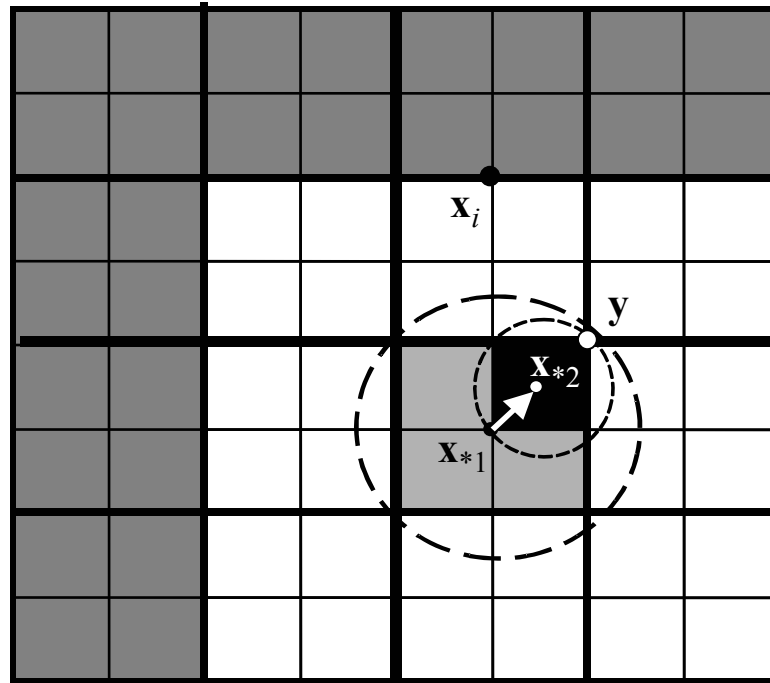
$$\begin{aligned}
 &= \frac{2}{2^{L-l} r (2 + \frac{\rho}{r})} \sum_{m=p}^{\infty} \frac{1}{(2 + \frac{\rho}{r})^m} \frac{1}{(1 + \frac{\rho}{r})^{m+1}} \left(2 + \frac{\rho}{r}\right)^{m+1} \\
 &= \frac{2}{2^{L-l} r} \sum_{m=p}^{\infty} \frac{1}{(1 + \frac{\rho}{r})^{m+1}} = \frac{2}{2^{L-l} r (1 + \frac{\rho}{r})^{p+1}} \sum_{m=0}^{\infty} \frac{1}{(1 + \frac{\rho}{r})^m} \\
 &= \frac{2}{2^{L-l} r (1 + \frac{\rho}{r})^{p+1}} \frac{1}{1 - \frac{1}{1 + \rho/r}} = \frac{2}{2^{L-l} r (1 + \frac{\rho}{r})^{p+1}} \frac{r}{\rho} \left(1 + \frac{\rho}{r}\right) \\
 &= \frac{2}{2^{L-l} \rho} \left(\frac{r}{r + \rho}\right)^p \leq \frac{2}{\rho} \left(\frac{r}{r + \rho}\right)^p.
 \end{aligned}$$

That's it!



$$\begin{aligned}
 d = 1: \quad &\rho / r = 2, \\
 d = 2: \quad &\rho / r = 2(2 - \sqrt{2}) / \sqrt{2} \\
 &= 2(\sqrt{2} - 1) > 0.8
 \end{aligned}$$

R|R-Translation Error



R|R-Translation Error(2)

$$\begin{aligned}
 & \left| \widehat{\Psi}_\alpha(\mathbf{y}, \mathbf{x}_k) - \widehat{\Psi}_{\alpha+1}(\mathbf{y}, \mathbf{x}_k) \right| \\
 &= \left| \sum_{n=0}^{p-1} \widehat{D}_n^{(\alpha)} R_n(\mathbf{y} - \mathbf{y}_*^{(\alpha)}) - \sum_{m=0}^{p-1} \widehat{D}_m^{(\alpha+1)} R_m(\mathbf{y} - \mathbf{y}_*^{(\alpha+1)}) \right| \\
 &= \left| \sum_{n=0}^{p-1} \sum_{m=0}^{\infty} (R|R)_{mn}(\mathbf{y}_*^{(\alpha+1)} - \mathbf{y}_*^{(\alpha)}) \widehat{D}_n^{(\alpha)} R_m(\mathbf{y} - \mathbf{y}_*^{(\alpha+1)}) \right. \\
 &\quad \left. - \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} (R|R)_{mn}(\mathbf{y}_*^{(\alpha+1)} - \mathbf{y}_*^{(\alpha)}) \widehat{D}_n^{(\alpha)} R_m(\mathbf{y} - \mathbf{y}_*^{(\alpha+1)}) \right| \\
 &= \left| \sum_{n=0}^{p-1} \sum_{m=p}^{\infty} (R|R)_{mn}(\mathbf{y}_*^{(\alpha+1)} - \mathbf{y}_*^{(\alpha)}) \widehat{D}_n^{(\alpha)} R_m(\mathbf{y} - \mathbf{y}_*^{(\alpha+1)}) \right| = 0.
 \end{aligned}$$

since $(R|R)_{mn}(t) = 0, \quad m > n$

This is something, but why?

R|R-Translation Error(3)

Indeed, in our case the regular basis functions are polynomials up to order $p-1$, which are obviously can be expressed via other polynomial basis up to order $p-1$ near arbitrary expansion center.

Zero error is provided due to domains of validity are includes hierarchically to larger validity domains.

Total Error

$$\begin{aligned} \text{AbsSingleSourceError} &\leq \text{MaxExpansionError}(p, R, r) + \text{MaxSRTranslationError}(p, r, \rho) \\ &= \left(\frac{r}{R}\right)^p \frac{1}{R-r} + \frac{2}{\rho} \left(\frac{r}{r+\rho}\right)^p. \end{aligned}$$

$$\begin{aligned} \text{AbsTotalError} &= N \cdot \text{AbsSingleSourceError} \\ &= N \left[\left(\frac{r}{R}\right)^p \frac{1}{R-r} + \frac{2}{\rho} \left(\frac{r}{r+\rho}\right)^p \right]. \end{aligned}$$

since $R > r + \rho$,

$$\text{AbsTotalError} < \frac{3N}{\rho} \left(\frac{r}{r+\rho}\right)^p.$$

Total Error(2)

$$d = 1 : \quad \rho = 2^{-L}, \quad r = 0.5 \cdot 2^{-L},$$

$$\text{AbsTotalError} < 3 \frac{2^L N}{3^p}.$$

$$d = 2 : \quad \rho = (2 - \sqrt{2})2^{-L}, \quad r = 0.5\sqrt{2} \cdot 2^{-L},$$

$$\begin{aligned} \text{AbsTotalError} &< 3 \frac{2^L N}{2 - \sqrt{2}} \left(\frac{\sqrt{2}}{2(\sqrt{2}/2 + 2 - \sqrt{2})} \right)^p \\ &= 3 \frac{2^L N}{2 - \sqrt{2}} \left(\frac{\sqrt{2}}{4 - \sqrt{2}} \right)^p < 5.2(0.6)^p \cdot 2^L N. \end{aligned}$$

Both formulae can be described as

$$\text{AbsTotalError} < C\alpha^p 2^L N = \epsilon(p, N, L, d).$$

Total Error(3)

$$AbsTotalError < C\alpha^p 2^L N = \epsilon(p, N, L, d).$$

Example: $N = 10^4$, $L = 10$, $d = 1$:

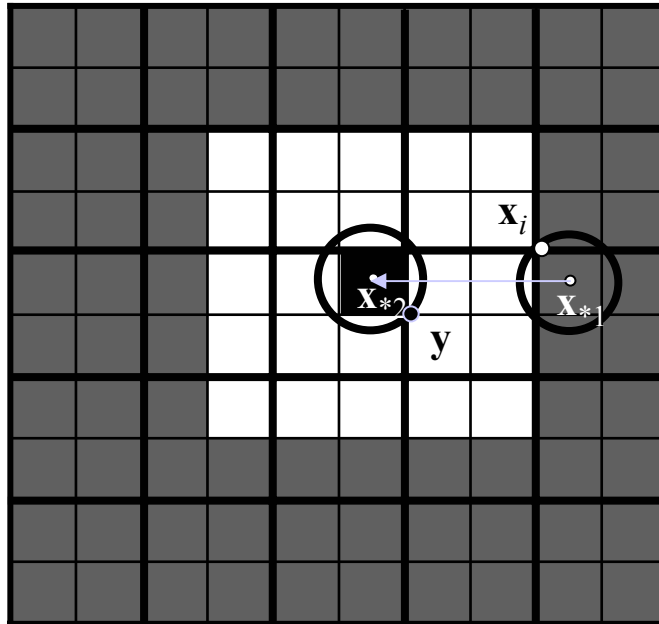
p	10	20	30	40
$\epsilon, <$	$6 \cdot 10^2$	$9 \cdot 10^{-3}$	$2 \cdot 10^{-7}$	$3 \cdot 10^{-12}$

Example: $N = 10^4$, $L = 10$, $d = 2$:

p	40	50	60	70
$\epsilon, <$	$8 \cdot 10^{-2}$	$5 \cdot 10^{-4}$	$3 \cdot 10^{-6}$	$2 \cdot 10^{-8}$

Total Error(4). 5-Neighborhood.

5-neighborhood



$$d = 1 : \quad \rho = 2 \cdot 2^{-L}, \quad r = 0.5 \cdot 2^{-L},$$

$$AbsTotalError < 3 \frac{2^L N}{5^p}.$$

$$d = 2 : \quad \rho = (3 - \sqrt{2}) 2^{-L}, \quad r = 0.5 \sqrt{2} \cdot 2^{-L},$$

$$\begin{aligned} AbsTotalError &< 3 \frac{2^L N}{3 - \sqrt{2}} \left(\frac{\sqrt{2}}{2(\sqrt{2}/2 + 3 - \sqrt{2})} \right)^p \\ &= 3 \frac{2^L N}{3 - \sqrt{2}} \left(\frac{\sqrt{2}}{6 - \sqrt{2}} \right)^p < 2(0.31)^p \cdot 2^L N. \end{aligned}$$

$$AbsTotalError < C\alpha^p 2^L N = \epsilon(p, N, L, d).$$

Total Error(5). 5-Neighborhood.

$$AbsTotalError < C\alpha^p 2^L N = \epsilon(p, N, L, d).$$

Example: $N = 10^4$, $L = 10$, $d = 1$:

p	10	15	20	25
$\epsilon, <$	4	$2 \cdot 10^{-3}$	$4 \cdot 10^{-7}$	$2 \cdot 10^{-10}$

Example: $N = 10^4$, $L = 10$, $d = 2$:

p	15	20	25	30
$\epsilon, <$	$5 \cdot 10^{-1}$	$2 \cdot 10^{-3}$	$4 \cdot 10^{-6}$	$2 \cdot 10^{-8}$

Optimization of MLFMM within error bounds

In the example considered, the FMM error depends on:

- Truncation number, p ;
- Max level of space subdivision, L ;
- Size of the neighborhood (3,5, or maybe other);
- Number of sources, N ;
- Problem dimensionality, d .

For fixed (given) N and d parameters p, L , and *Neighborhood Size* can be optimized.

MLFMM Complexity

$$Cost_{MLFMM} = (M + N)P + 2^d(P_4(d) + 2) \frac{N}{s} Cost_{Translation}(P) + P_2(d)sM Cost_{Func},$$

$$N = 2^{L_*d}, \quad s = 2^{L_s d}, \quad L = L_* - L_s = \frac{1}{d} \log \frac{N}{s}.$$

Consider

$$\epsilon(p, N, L, d) = C\alpha^p 2^L N = C\alpha^p \left(\frac{N}{s}\right)^{1/d} N < \epsilon_0$$

$$p > \frac{1}{\log \frac{1}{\alpha}} \log \frac{CN^{1+1/d}}{\epsilon_0 s^{1/d}} \sim a \log N - b \log s + c$$

$$Cost_{MLFMM}(s) = (M + N)p(N, s) + 2^d(P_4(d) + 2) \frac{N}{s} p^2(N, s) + P_2(d)sM$$

$$0 = \frac{d Cost_{MLFMM}(s)}{ds} \sim -b \frac{M + N}{s} - 2^d(P_4(d) + 2) \frac{N}{s^2} (a \log N - b \log s + c)^2$$

$$- 2^{d+1}(P_4(d) + 2) \frac{bN}{s^2} (a \log N - b \log s + c) + P_2(d)M$$

MLFMM Complexity(2)

For asymptotic optimum s we can consider $a \log N - b \log s \sim a \log N$, so

$$s_{opt} \sim \left[\frac{2^d (P_4(d) + 2) a^2 \log N}{P_2(d) M} \right]^{1/2} \sim \left[\frac{N \log^2 N}{M} \right]^{1/2} \sim \log N.$$

$$Cost_{MLFMM_{opt}} \sim O(N \log N).$$