# MAIT 627 Fast Multipole Methods

## Lecture 17

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# Outline

- Asymptotically faster translation methods
  - Use of the FFT: 2D Laplace: $O(p\log p)$, 3D Laplace: $O(p^2\log p)$
  - RCR-decomposition for 3D Laplace: $O(p^3)$
  - Reduction of asymptotic constants: Exponential forms for 2D: $O(p^2)$ and 3D: $O(p^3)$

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# Fast Toeplitz matrix-vector multiplication(1)

$$B_m = \sum_{n=0}^{N} W_{mn} A_n, \quad m = 0,\ldots,M$$

$$W_{mn} = U_{m-n}.$$

$$\mathbf{W} = \{W_{mn}\} = \begin{pmatrix} W_{00} & W_{01} & \ldots & W_{0N} \\ W_{10} & W_{11} & \ldots & W_{1N} \\ \ldots & \ldots & \ldots & \ldots \\ W_{M0} & W_{M1} & \ldots & W_{MN} \end{pmatrix} = \begin{pmatrix} U_0 & U_{-1} & \ldots & U_{-N} \\ U_1 & U_0 & \ldots & U_{1-N} \\ \ldots & \ldots & \ldots & \ldots \\ U_M & U_{M-1} & \ldots & U_{M-N} \end{pmatrix}.$$

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# Fast Toeplitz matrix-vector multiplication(2)

$$u(t) = \sum_{l=-N}^{M} U_l e^{ilt}, \quad b(t) = \sum_{m=0}^{M} B_m e^{imt}, \quad a(t) = \sum_{n=0}^{N} A_n e^{int}.$$

$$a(t)u(t) = \sum_{n=0}^{N} A_n e^{int} \sum_{l=-N}^{M} U_l e^{ilt} = \sum_{n=0}^{N}\sum_{l=-N}^{M} A_n U_l e^{i(n+l)t}$$

$$= \sum_{m=-N}^{-1} e^{imt} \sum_{n=0}^{N+m} A_n U_{m-n} + \sum_{m=0}^{M} e^{imt} \sum_{n=0}^{N} A_n U_{m+n} + \sum_{m=M+1}^{M+N} e^{imt} \sum_{n=m-M-1}^{N} A_n U_{m-n}$$

$$= \sum_{m=-N}^{-1} e^{imt} \sum_{n=0}^{N+m} A_n U_{m-n} + b(t) + \sum_{m=M+1}^{M+N} e^{imt} \sum_{n=m-M-1}^{N} A_n U_{m-n}.$$

So to get $b(t)$ we should perform inverse FFT of $\{A_n\}, \{U_l\}$, multiply $a(t)u(t)$, perform the forward FFT, and then take take harmonics from $0$ to $M$ which are $\{B_m\}$.

# Fast Hankel matrix-vector multiplication(1)

$$B_m = \sum_{n=0}^{N} W_{mn} A_n, \quad m = 0, \ldots, N$$

$$W_{mn} = U_{m+n}.$$

$$\mathbf{W} = \{W_{mn}\} = \begin{pmatrix} W_{00} & W_{01} & \ldots & W_{0N} \\ W_{10} & W_{11} & \ldots & W_{1N} \\ \ldots & \ldots & \ldots & \ldots \\ W_{N0} & W_{N1} & \ldots & W_{NN} \end{pmatrix} = \begin{pmatrix} U_0 & U_1 & \ldots & U_N \\ U_1 & U_2 & \ldots & U_{N+1} \\ \ldots & \ldots & \ldots & \ldots \\ U_N & U_{N+1} & \ldots & U_{2N} \end{pmatrix}$$

# Fast Hankel matrix-vector multiplication(2)

$$u(t) = \sum_{l=0}^{2N} U_l e^{ilt}, \quad b(t) = \sum_{m=0}^{N} B_m e^{imt}, \quad a(t) = \sum_{n=0}^{N} \overline{A_n} e^{int}.$$

$$\begin{aligned} \overline{a(t)}u(t) &= \sum_{n=0}^{N} A_n e^{-int} \sum_{l=0}^{2N} U_l e^{ilt} = \sum_{n=0}^{N} \sum_{l=0}^{2N} A_n U_l e^{i(l-n)t} \\ &= \sum_{m=-N}^{2N} e^{imt} \sum_{n=0}^{N} A_n U_{m+n} \\ &= \sum_{m=-N}^{-1} e^{imt} \sum_{n=0}^{N} A_n U_{m+n} + \sum_{m=0}^{N} e^{imt} \sum_{n=0}^{N} A_n U_{m+n} + \sum_{m=N+1}^{2N} e^{imt} \sum_{n=0}^{N} A_n U_{m+n} \\ &= \sum_{m=-N}^{-1} e^{imt} \sum_{n=0}^{N} A_n U_{m+n} + b(t) + \sum_{m=N+1}^{2N} e^{imt} \sum_{n=0}^{N} A_n U_{m+n}. \end{aligned}$$

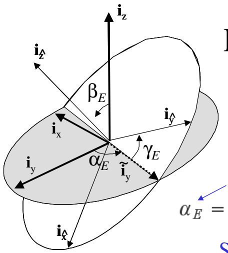
So to get $b(t)$ we should perform inverse FFT of $\{\overline{A_n}\}$, $\{U_l\}$, multiply $\overline{a(t)}u(t)$, perform the forward FFT, and then take take harmonics from $0$ to $M$ which are $\{B_m\}$.

# Use of the FFT in the FMM

- 1D Toeplitz-Hankel structure of translation operators for 2D Laplace; 2D Toeplitz-Hankel structure for 3D Laplace (convolution should be properly modified, e.g. see **W.D. Elliott & J.A. Board, Jr.:** ``Fast Fourier Transform Accelerated Fast Multipole Algorithm", *SIAM J. Sci. Comput.* Vol. 17, No. 2, pp. 398-415, 1996).
- Doing straightforward for 2D Laplace, one needs 3FFTs per translation (27*3=81 FFTs for S|R-translation in the E4-neighborhood);
- How to save on FFTs:
  - Transform translation operators into Fourier domain and save (one time precomputation step); This cost can be dropped from operations count as it can be precomputed and stored forever);
  - For translation: Transform coefficients into Fourier domain ($O(p\log p)$) for each box from which translation should be performed (27 FFTs);
  - Perform point-by-point matrix vector multiplication for each translation (27$p$);
  - Consolidate transforms in the Fourier domain (27$p$);
  - Pad with zeros (27$p$);
  - Perform one Fourier transform for the translated box (1 FFT);
  - Total: 28 FFTs (about 1 FFT per translation);

# RCR-decomposition (**R**otation- **C**oaxial translation- **R**otation) for 3D Laplace

# Rotations of coordinates



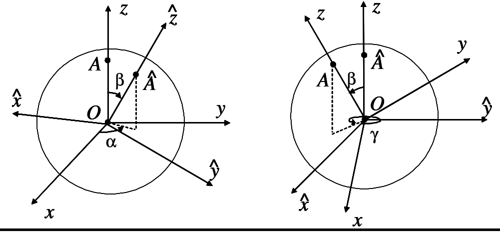
Rotation Matrix

$$Q = \begin{bmatrix} \mathbf{i}_{\hat{x}}\cdot\mathbf{i}_x & \mathbf{i}_{\hat{x}}\cdot\mathbf{i}_y & \mathbf{i}_{\hat{x}}\cdot\mathbf{i}_z \\ \mathbf{i}_{\hat{y}}\cdot\mathbf{i}_x & \mathbf{i}_{\hat{y}}\cdot\mathbf{i}_y & \mathbf{i}_{\hat{y}}\cdot\mathbf{i}_z \\ \mathbf{i}_{\hat{z}}\cdot\mathbf{i}_x & \mathbf{i}_{\hat{z}}\cdot\mathbf{i}_y & \mathbf{i}_{\hat{z}}\cdot\mathbf{i}_z \end{bmatrix}$$

Euler Angles

$$\alpha_E = \alpha, \quad \beta_E = \beta, \quad \gamma_E = \pi - \gamma.$$

Spherical Polar Angles



# Rotations of elementary solutions of the 3D Laplace equation

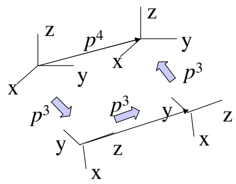
Rotations

$$Y_n^m(\theta,\varphi) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) Y_n^{\nu}\left(\hat{\theta},\hat{\varphi}\right),$$

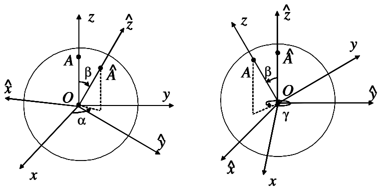
$$S_n^m(\mathbf{r}_p) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) S_n^{\nu}(\hat{\mathbf{r}}_p), \qquad |\hat{\mathbf{r}}_p| = |\mathbf{r}_p|,$$

$$R_n^m(\mathbf{r}_p) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) R_n^{\nu}(\hat{\mathbf{r}}_p), \qquad |\hat{\mathbf{r}}_p| = |\mathbf{r}_p|,$$

# Rotation-Coaxial Translation Decomposition



Rotation



Coaxial Translation

$$S_n^m(\tilde{\mathbf{r}} + \mathbf{i}_z d) = \sum_{l=|m|}^{\infty} (S|R)_{ln}^m(d) R_l^m(\tilde{\mathbf{r}}), \qquad |\tilde{\mathbf{r}}| < d,$$

$$S_n^m(\tilde{\mathbf{r}} + \mathbf{i}_z d) = \sum_{l=|m|}^{\infty} (S|S)_{ln}^m(d) S_l^m(\tilde{\mathbf{r}}), \qquad |\tilde{\mathbf{r}}| > d,$$

$$R_n^m(\tilde{\mathbf{r}} + \mathbf{i}_z d) = \sum_{l=|m|}^{\infty} (R|R)_{ln}^m(d) R_l^m(\tilde{\mathbf{r}}).$$

$$(E|F)_{ln}^m(d) = (E|F)_{ln}^{mm}(\mathbf{d})\big|_{\theta'_{pq}=0}, \quad E,F = S,R.$$

$$Y_n^m(\theta,\varphi) = \sum_{\nu=-n}^{n} T_n^{\nu m}(Q) Y_n^{\nu}\left(\hat{\theta},\hat{\varphi}\right),$$

$$Q = \begin{bmatrix} \mathbf{i}_{\hat{x}}\cdot\mathbf{i}_x & \mathbf{i}_{\hat{x}}\cdot\mathbf{i}_y & \mathbf{i}_{\hat{x}}\cdot\mathbf{i}_z \\ \mathbf{i}_{\hat{y}}\cdot\mathbf{i}_x & \mathbf{i}_{\hat{y}}\cdot\mathbf{i}_y & \mathbf{i}_{\hat{y}}\cdot\mathbf{i}_z \\ \mathbf{i}_{\hat{z}}\cdot\mathbf{i}_x & \mathbf{i}_{\hat{z}}\cdot\mathbf{i}_y & \mathbf{i}_{\hat{z}}\cdot\mathbf{i}_z \end{bmatrix}$$

# Decomposition into Subspaces

$$\Phi^p(\mathbf{r}) = \sum_{n=0}^{p}\sum_{m=-n}^{n} A_n^m F_n^m(\mathbf{r}) = \sum_{m=-p}^{p}\sum_{n=|m|}^{p} A_n^m F_n^m(\mathbf{r}) = \mathbf{A}\cdot\mathbf{F}, \quad F = S,R.$$

$$\mathbf{A} = \mathbf{A}^0 \oplus \mathbf{A}^{\pm 1} \oplus \ldots = \sum_{m=-\infty}^{\infty} \oplus \mathbf{A}^m,$$

where

$$\mathbf{A}^m = \left(A_{|m|}^m, A_{|m|+1}^m, A_{|m|+2}^m, \ldots\right)^T, \quad m = 0, \pm 1, \pm 2, \ldots,$$

and as the direct sum of finite blocks $\mathbf{A}_n$ corresponding to degree $m$ :

$$\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1 \oplus \ldots = \sum_{n=0}^{\infty} \oplus \mathbf{A}_n,$$

where

$$\mathbf{A}_n = (A_n^{-n}, \ldots, A_n^n)^T, \quad n = 0, 1, 2, \ldots.$$

## So the coaxial translation operator has invariant subspaces at fixed order, *m*, while the rotation operator has invariant subspaces at fixed degree, *n*.

Coaxial Translation:

$$(\mathbf{S}|\mathbf{R})=(\mathbf{S}|\mathbf{R})^{0}\oplus(\mathbf{S}|\mathbf{R})^{\pm1}\oplus\ldots=\sum_{m=-\infty}^{\infty}\oplus(\mathbf{S}|\mathbf{R})^{m},$$

Rotation

$$(\mathbf{S}|\mathbf{R})=(\mathbf{S}|\mathbf{R})_{0}\oplus(\mathbf{S}|\mathbf{R})_{1}\oplus\ldots=\sum_{n=0}^{\infty}\oplus(\mathbf{S}|\mathbf{R})_{n},$$

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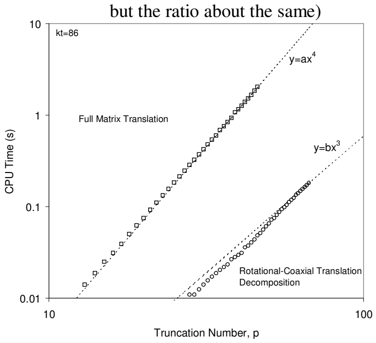
## Comparison of Direct Matrix Translation and Coaxial Translation-Rotation Decomposition

(Figure for the Helmholtz equation; for the Laplace CPU time is different, but the ratio about the same)



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## Exponential form for the S|R-translation ("Plane wave expansion").

L. Greengard and V. Rokhlin, A new version of the fast multipole method for the Laplace equation in three dimensions, Acta Numerica, 6, 1997, 229-269.

H. Cheng, L. Greengard, and V. Rokhlin, A fast adaptive multipole algorithm in three dimensions, J. Comput. Phys., 155, 1999, 468-498.

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## Expansions of the Greens function and arbitrary harmonic function

$$G(\mathbf{r},\mathbf{r}_0)=\frac{1}{8\pi^2}\int_0^{\infty}e^{-\lambda(z-z_0)}\int_0^{2\pi}e^{i\lambda[(x-x_0)\cos\alpha+(y-y_0)\sin\alpha]}d\alpha d\lambda,$$

$$\mathbf{r}=(x,y,z),\quad \mathbf{r}_0=(x_0,y_0,z_0),\quad z>z_0.$$

For

$$a\leqslant z-z_0\leqslant b,\ \sqrt{(x-x_0)^2+(y-y_0)^2}\leqslant c,\quad (a=1,b=4,c=4\sqrt{2}):$$

$$G(\mathbf{r},\mathbf{r}_0)=\frac{1}{4\pi}\sum_{k=1}^{S(\epsilon)}\frac{w_k}{M_k}e^{-\lambda_k(z-z_0)}\sum_{j=1}^{M_k}e^{i\lambda_k[(x-x_0)\cos\alpha_{jk}+(y-y_0)\sin\alpha_{jk}]},$$

$$\phi(\mathbf{r})=\sum_{k=1}^{S(\epsilon)}\sum_{j=1}^{M_k}W_{kj}e^{-\lambda_k z}e^{i\lambda_k(x\cos\alpha_{jk}+y\sin\alpha_{jk})},$$

$$S_{\exp}=\sum_{k=1}^{S(\epsilon)}M_k=\sigma p^2,\quad S(\epsilon)=\kappa p.$$

## S|R-translation

$$\phi(\mathbf{r}+\mathbf{t}) = \sum_{k=1}^{S(\epsilon)}\sum_{j=1}^{M_k} W_{kj} e^{-\lambda_k t_z} e^{i\lambda_k (t_x \cos\alpha_{jk} + t_y \sin\alpha_{jk})} e^{-\lambda_k z} e^{i\lambda_k (x\cos\alpha_{jk} + y\sin\alpha_{jk})}$$

$$= \sum_{k=1}^{S(\epsilon)}\sum_{j=1}^{M_k} \widehat{W}_{kj} e^{-\lambda_k z} e^{i\lambda_k (x\cos\alpha_{jk} + y\sin\alpha_{jk})} = \hat{\phi}(\mathbf{r}),$$

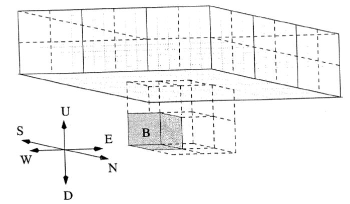
$$\widehat{W}_{kj} = E_{kj} W_{kj}, \quad E_{kj} = e^{-\lambda_k t_z} e^{i\lambda_k (t_x \cos\alpha_{jk} + t_y \sin\alpha_{jk})}, \quad \mathbf{t} = (t_x, t_y, t_z).$$

$$W_{kj} = \frac{w_k}{M_k d} \sum_{m=-(p-1)}^{p-1} e^{im\alpha_{jk}} \sum_{n=|m|}^{p-1} \frac{1}{\beta_n^m \beta_{(1)n}^m} \left(\frac{\lambda_k}{d}\right)^n \phi_n^m, \quad k = 1,..,s(\epsilon), \quad j = 1,\ldots,M_k,$$

$$\phi_n^m = \alpha_n^m \alpha_{(1)n}^m \sum_{k=1}^{S(\epsilon)} \left(\frac{\lambda_k}{d}\right)^n \sum_{j=1}^{M_k} W_{kj} e^{-im\alpha_{jk}},$$

$$N^{(E|R)} = N^{(S|E)} \approx (\kappa + 4\sigma)p^3.$$

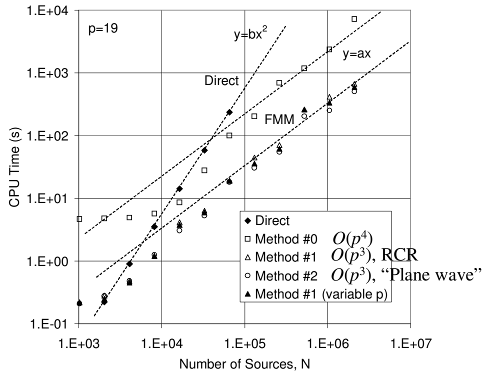
## Translation scheme



Needs also 6 translation directions;
Performed by rotation transforms (axis flips);
$O(p^3)$ complexity;

(From Greengard & Rokhlin, 1997)

## Comparison of translation methods (for the same accuracy)



## Theory of Signature Function

# Translation Kernels

$$\Lambda_r(z;\alpha) = \sum_{n=0}^{\infty} e^{in\alpha} R_n(z),$$

$$\Lambda_s^{(p)}(z;\alpha) = \sum_{n=0}^{p-1} e^{-in\alpha} S_n(z).$$

$$\Lambda_r(z;\alpha) = \sum_{n=0}^{\infty} e^{in\alpha} R_n(z) = \sum_{n=0}^{\infty} \frac{(-ze^{i\alpha})^n}{n!} = e^{-ze^{i\alpha}}.$$

# Integral representation of basis functions(1)

$$R_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}\Lambda_r(z;\alpha)d\alpha, \quad n = 0,1,\ldots, \quad 0 \leqslant |z| < \infty.$$

Indeed:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}\Lambda_r(z;\alpha)d\alpha &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}\sum_{m=0}^{\infty} e^{im\alpha}R_m(z)d\alpha \\ &= \sum_{m=0}^{\infty} R_m(z)\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\alpha}d\alpha \\ &= \sum_{m=0}^{\infty} R_m(z)\delta_{mn} = R_n(z).\end{aligned}$$

# Integral representation of basis functions(2)

$$S_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\alpha}\Lambda_s^{(p)}(z;\alpha)d\alpha, \quad n = 0,1,\ldots,p-1, \quad 0 < |z| < \infty.$$

Indeed:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{in\alpha}\Lambda_s^{(p)}(z;\alpha)d\alpha &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\alpha}\sum_{m=0}^{p-1} e^{-im\alpha}S_m(z)d\alpha \\ &= \sum_{m=0}^{p-1} S_m(z)\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\alpha}d\alpha \\ &= \sum_{m=0}^{p-1} S_m(z)\delta_{mn} = S_n(z).\end{aligned}$$

# R-signature function

# Theory of Signature Function
2D case

# Translation kernels

$$\Lambda_r(z;\alpha) = \sum_{n=0}^{\infty} e^{in\alpha} R_n(z),$$

$$\Lambda_s^{(p)}(z;\alpha) = \sum_{n=0}^{p-1} e^{-in\alpha} S_n(z).$$

$$\Lambda_r(z;\alpha) = \sum_{n=0}^{\infty} e^{in\alpha} R_n(z) = \sum_{n=0}^{\infty} \frac{(-ze^{i\alpha})^n}{n!} = e^{-ze^{i\alpha}}.$$

# Integral representation of basis functions(1)

$$R_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}\Lambda_r(z;\alpha)d\alpha, \quad n = 0,1,\ldots, \quad 0 \leqslant |z| < \infty.$$

Indeed:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}\Lambda_r(z;\alpha)d\alpha &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}\sum_{m=0}^{\infty} e^{im\alpha}R_m(z)d\alpha \\ &= \sum_{m=0}^{\infty} R_m(z)\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\alpha}d\alpha \\ &= \sum_{m=0}^{\infty} R_m(z)\delta_{mn} = R_n(z).\end{aligned}$$

# Integral representation of basis functions(2)

$$S_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\alpha}\Lambda_s^{(p)}(z;\alpha)d\alpha, \quad n = 0,1,\ldots,p-1, \quad 0 < |z| < \infty.$$

Indeed:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{in\alpha}\Lambda_s^{(p)}(z;\alpha)d\alpha &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\alpha}\sum_{m=0}^{p-1} e^{-im\alpha}S_m(z)d\alpha \\ &= \sum_{m=0}^{p-1} S_m(z)\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\alpha}d\alpha \\ &= \sum_{m=0}^{p-1} S_m(z)\delta_{mn} = S_n(z).\end{aligned}$$

# R-signature function

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n R_n(z) \to \quad \Phi_r(\alpha) = \sum_{n=0}^{\infty} \phi_n e^{-in\alpha}.$$

Arbitrary function

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n R_n(z) = \sum_{n=0}^{\infty} \phi_n \frac{1}{2\pi} \int_0^{2\pi} e^{-in\alpha} \Lambda_r(z;\alpha) d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} \phi_n e^{-in\alpha} \right) \Lambda_r(z;\alpha) d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \Phi_r(\alpha) \Lambda_r(z;\alpha) d\alpha.$$

Green's function:

$$G(z - z_0) = \lim_{p\to\infty} \sum_{n=0}^{p-1} S_n(-z_0) R_n(z) =$$

$$= \lim_{p\to\infty} \sum_{n=0}^{p-1} S_n(-z_0) \frac{1}{2\pi} \int_0^{2\pi} e^{-in\alpha} \Lambda_r(z;\alpha) d\alpha$$

$$= \lim_{p\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \Lambda_s^{(p)}(-z_0;\alpha) \Lambda_r(z;\alpha) d\alpha, \quad 0 \leqslant |z| < a < |z_0|.$$

# S-signature function

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n S_n(z) \to \quad \Phi_s(\alpha) = \sum_{n=0}^{\infty} \phi_n e^{in\alpha}.$$

Arbitrary function

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n S_n(z) = \lim_{p\to\infty} \sum_{n=0}^{p-1} \phi_n \frac{1}{2\pi} \int_0^{2\pi} e^{in\alpha} \Lambda_s^{(p)}(z;\alpha) d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} \phi_n e^{in\alpha} \right) \Lambda_s^{(p)}(z;\alpha) d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \Phi_s(\alpha) \Lambda_s^{(p)}(z;\alpha) d\alpha.$$

$$\phi(z) = \lim_{p\to\infty} \phi^{(p)}(z), \quad \phi^{(p)}(z) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_s(\alpha) \Lambda_s^{(p)}(z;\alpha) d\alpha, \quad 0 < a < |z|,$$

Green's function:

$$G(z - z_0) = \ln \frac{1}{z - z_0} = \sum_{n=0}^{\infty} R_n(-z_0) S_n(z), \quad |z| > |z_0|,$$

$$G(z - z_0) \to \Phi_s(\alpha) = \Lambda_r(-z_0;\alpha).$$

# Translations of signature function

R|R-translation

$$\hat{\phi}(z) = \phi(z + t) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_r(\alpha) \Lambda_r(z + t;\alpha) d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \Phi_r(\alpha) \Lambda_r(t;\alpha) \Lambda_r(z;\alpha) d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \widehat{\Phi}_r(\alpha) \Lambda_r(z;\alpha) d\alpha,$$

since

$$\Lambda_r(z + t;\alpha) = e^{-(z+t)e^{i\alpha}} = e^{-te^{i\alpha}} e^{-ze^{i\alpha}} = \Lambda_r(t;\alpha) \Lambda_r(z;\alpha).$$

So

$$\widehat{\Phi}_r(\alpha) = (\mathcal{R}|\mathcal{R})(t)[\Phi_r(\alpha)] = \Lambda_r(t;\alpha) \Phi_r(\alpha).$$

# Translations of signature function

S|S-translation

$$\hat{\phi}_n = \sum_{n'=0}^{\infty} (S|S)_{nn'} \phi_{n'} = \sum_{n'=0}^{\infty} R_{n-n'}(t) \phi_{n'}.$$

$$\widehat{\Phi}_s(\alpha) = \sum_{n=0}^{\infty} \hat{\phi}_n e^{in\alpha} = \sum_{n'=0}^{\infty} \left[ \sum_{n=0}^{\infty} e^{i(n-n')\alpha} R_{n-n'}(t) \right] e^{in'\alpha} \phi_{n'}$$

$$= \Lambda_r(t;\alpha) \sum_{n'=0}^{\infty} e^{in'\alpha} \phi_{n'} = \Lambda_r(t;\alpha) \Phi_s(\beta, \alpha).$$

$$\widehat{\Phi}_s(\alpha) = (\mathcal{S}|\mathcal{S})(\mathbf{t})[\Phi_s(\alpha)] = \Lambda_r(t;\alpha) \Phi_s(\alpha).$$

# Translations of signature function

S|S-translation / Corollary

$$\tilde{\phi}(z) = \phi(z+t) = \lim_{p\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(\alpha)\Lambda_s^{(p)}(z+t;\alpha)d\alpha$$

$$= \lim_{p\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(\alpha)\Lambda_s^{(p)}(z;\alpha)\Lambda_r(t;\alpha)d\alpha, \quad |z| > |t|.$$

# Translations of signature function

S|R-translation

Corollary from S|S:

$$\tilde{\phi}(z) = \phi(z+t) = \lim_{p\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(\alpha)\Lambda_s^{(p)}(z+t;\alpha)d\alpha$$

$$= \lim_{p\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(\alpha)\Lambda_s^{(p)}(z;\alpha)\Lambda_r(t;\alpha)d\alpha, \quad |z| > |t|.$$

Exchange *z* and *t*:

$$\tilde{\phi}(z) = \phi(z+t) = \lim_{p\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(\alpha)\Lambda_s^{(p)}(z+t;\alpha)d\alpha$$

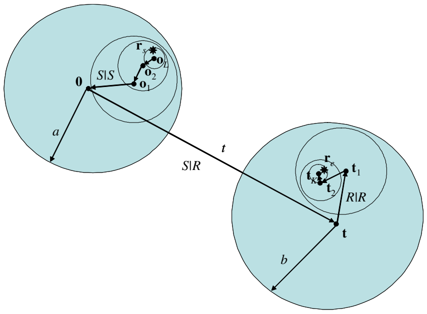
$$= \lim_{p\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi_s(\alpha)\Lambda_s^{(p)}(t;\alpha)\Lambda_r(z;\alpha)d\alpha, \quad |t| > |z|.$$

S|R:

$$\widehat{\Phi}_r^{(p)}(\alpha) = (\mathcal{S}|\mathcal{R})^{(p)}(t)[\Phi_s(\alpha)] = \Lambda_s^{(p)}(t;\alpha)\Phi_s(\alpha).$$

$$\phi(z) = \lim_{p\to\infty}\phi^{(p)}(z), \quad \tilde{\phi}^{(p)}(z) = \frac{1}{2\pi}\int_0^{2\pi} \widehat{\Phi}_r^{(p)}(\alpha)\Lambda_r(z;\alpha)d\alpha.$$

# Translation error



# Translation error

$$\Phi_s(\alpha) = \Lambda_r(\mathbf{o}_L - \mathbf{r}_s;\alpha)$$

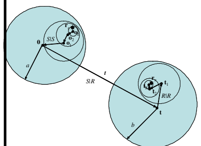
$$\Phi_s'(\alpha) = \Lambda_r(-\mathbf{o}_1;\alpha)\Lambda_r(\mathbf{o}_1 - \mathbf{o}_2;\alpha)\ldots\Lambda_r(\mathbf{o}_{L-1} - \mathbf{o}_L;\alpha)\Phi_s(\alpha),$$

$$\Phi_r'(\alpha) = \Lambda_s^{(p)}(\mathbf{t};\alpha)\Phi_s'(\alpha),$$

← The only place for error!

$$\Phi_r(\alpha) = \Lambda_r(\mathbf{t}_K - \mathbf{t}_{K-1};\alpha)\Lambda_r(\mathbf{t}_2 - \mathbf{t}_1;\alpha)\ldots\Lambda_r(\mathbf{t}_1 - \mathbf{t};\alpha)\Phi_r'(\alpha),$$

$$\Psi(\alpha) = \Lambda_r(\mathbf{r}_e - \mathbf{t}_K;\alpha)\Phi_r(\alpha)$$



$$\Phi_s'(\alpha) = \Lambda_r(-\mathbf{o}_L;\alpha)\Phi_s(\alpha),$$

$$\Phi_s'(\alpha) = \Lambda_r(-\mathbf{r}_s;\alpha).$$

$$\Phi_r(\alpha) = \Lambda_r(\mathbf{t}_K - \mathbf{t};\alpha)\Phi_r'(\alpha).$$

$$\Psi(\alpha) = \Lambda_r(\mathbf{r}_e - \mathbf{t}_K;\alpha)\Phi_r(\alpha) = \Lambda_r(\mathbf{r}_e - \mathbf{t};\alpha)\Phi_r'(\alpha).$$

# Sampling of signature function

The bandwidth of the singular kernel can differ from the number of samples!

$$\phi^{(p)}(z) = \frac{1}{2\pi}\int_0^{2\pi}\Psi(\alpha)d\alpha \approx \frac{1}{p'}\sum_{k=0}^{p'-1}\Lambda_r(z_e - z_s - t, \alpha_k)\Lambda_s^{(p)}(z_*, \alpha_k).$$

$$\alpha_k = k\gamma_{p'}, \quad k = 0, \ldots, p'-1, \quad \gamma_{p'} = \frac{2\pi}{p'}.$$

# Error bound

$$\left|\epsilon_1^{(p)}\right| = \left|\phi(z_e) - \phi_1^{(p)}(z_e)\right| \leqslant \sum_{n=p}^{\infty}\frac{(n-1)!}{t^n n!}\left(\frac{r}{t}\right)^n < \left(\frac{r}{t}\right)^p\left[\frac{1}{p} + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{t}\right)^n\right]$$
$$= \left(\frac{r}{t}\right)^p\left[\frac{1}{p} - \ln\left(1 - \frac{r}{t}\right)\right] \leqslant \left(\frac{2a}{t}\right)^p\left[\frac{1}{p} - \ln\left(1 - \frac{2a}{t}\right)\right].$$

$$|\epsilon^{(p)}| \leqslant \left(\frac{a}{t-a}\right)^p\left[\frac{1}{p} - \ln\left(1 - \frac{a}{t-a}\right)\right].$$

$$\frac{\frac{2a}{t}}{\frac{a}{t-a}} = \frac{2t-2a}{t} = 2 - \frac{2a}{t} > 1.$$

$$\left(\frac{2a}{t}\right)^{p_{\mathit{eff}}} = \left(\frac{a}{t-a}\right)^p,$$

$$p_{\mathit{eff}} = p\frac{\ln\left(\frac{a}{t-a}\right)}{\ln\left(\frac{2a}{t}\right)} = p\frac{\ln\left(\frac{\sqrt{2}/2}{2-\sqrt{2}/2}\right)}{\ln\left(\frac{2\left(\sqrt{2}/2\right)}{2}\right)} \approx 1.74p.$$

# Exponential form for the S|R-translation (“Plane wave expansion”), 2D case

Tomasz Hrycak and Vladimir Rokhlin.
An improved fast multipole algorithm for potential fields.
*SIAM Journal of Scientific Computing*, 19(6):1804-1826, 1998.

# Exponential expansions

$$\frac{1}{z - z_0} = \int_0^{\infty} e^{-x(z-z_0)}dx \approx \sum_{j=1}^{S(\epsilon)} w_j e^{-x_j(z-z_0)}, \quad \mathrm{Re}(z - z_0) > 0.$$

$$\phi(z) = \sum_{k=1}^{M}\frac{q_k}{z - z_k} \approx \sum_{j=1}^{S(\epsilon)}\sum_{k=1}^{M} w_j q_k e^{-x_j(z-z_k)} = \sum_{j=1}^{S(\epsilon)} W_j e^{-x_j z},$$

$$W_j = w_j\sum_{k=1}^{M} q_k e^{x_j z_k}.$$

$$\phi(z+t) = \sum_{j=1}^{S(\epsilon)} W_j e^{-x_j(z+t)} = \sum_{j=1}^{S(\epsilon)} E_j W_j e^{-x_j z} = \sum_{j=1}^{S(\epsilon)}\widehat{E}_j e^{-x_j z} = \widehat{\phi}(z),$$

$$\widehat{E}_j = E_j W_j, \quad E_j = e^{-x_j t}.$$

# Conversion of expansions

$$\phi(z) = \sum_{n=0}^{p-1} c_n S_n(z) = \sum_{j=1}^{S(\epsilon)} W_j e^{-x_j z}, \quad W_j = \sum_{n=0}^{p-1} (S|E)_{jn} c_n.$$

$$\phi(z) = \sum_{j=1}^{S(\epsilon)} W_j e^{-x_j z} = \sum_{n=0}^{p-1} d_n R_n(z), \quad d_n = \sum_{j=1}^{S(\epsilon)} (E|R)_{nj} W_j.$$

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# S|R-translation scheme

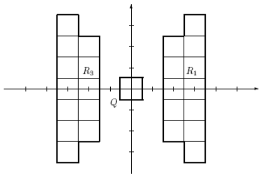

Fig. 1. *The domains $R_1$ and $R_3$.*

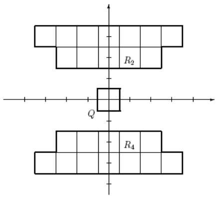

Fig. 2. *The domains $R_2$ and $R_4$.*