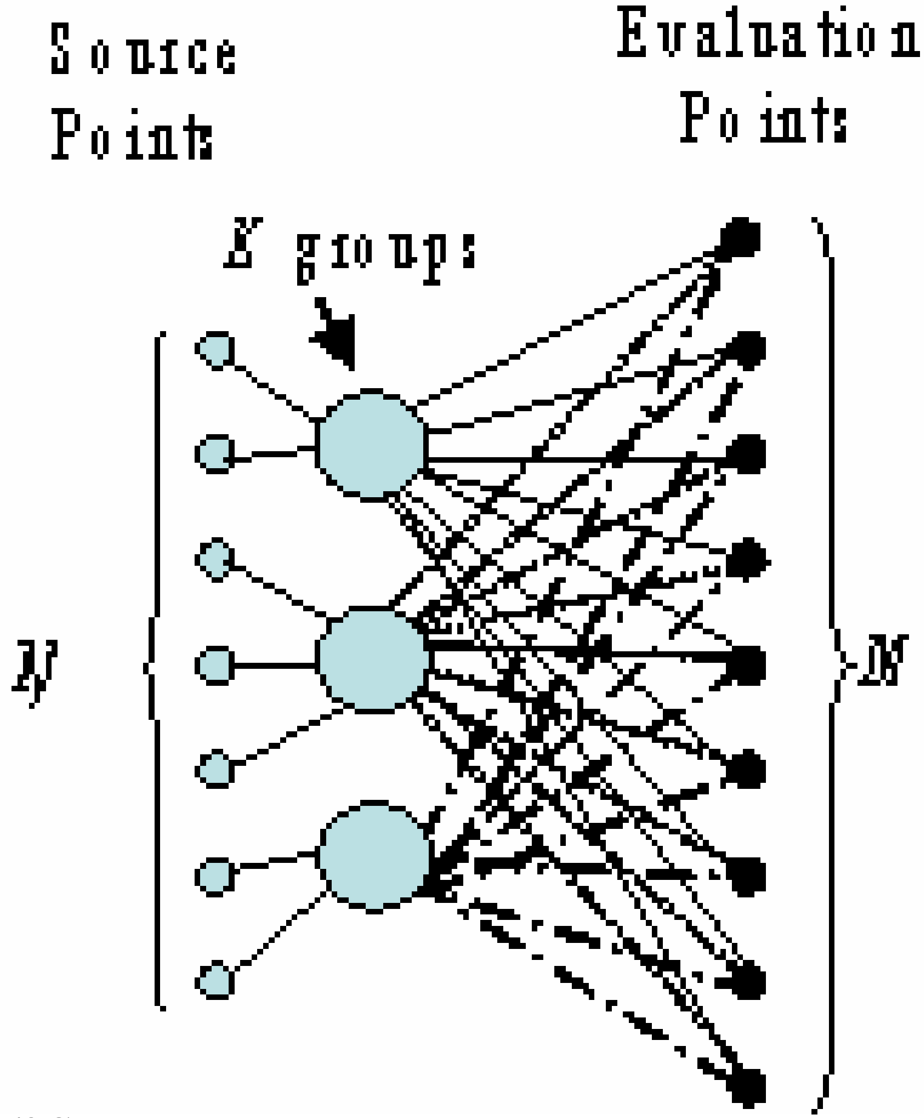
Lecture 17:CMSC 878R/AMSC698R

Exam – Q3

- Question 3
- N sources
- Need to form *S* expansion for each about the center of one of the *K* boxes
- Cost is *NP*
- Each box has *N/K* sources
- Need to sum up the *K* expansions
- Evaluate these at *M* points
- So we have *KMP* operations



Problem 3

- Take $M = N$
- In the neighboring boxes we have to direct evaluation (as the S expansion is not valid)
- There are $3^d - 1$ neighbors for each of the K boxes
- Each box contains N/K sources and M/K evaluation points
- Direct evaluation requires (MN/K^2) for one pair
- Total $(3^d - 1)(MN/K^2)K$
- Putting it together we have
- $F(K,N) = NP + (3^d - 1)N^2/K + KNP$

First term always smaller than 3rd

$dF/dK = -(3^d - 1)N^2/K^2 + NP = 0$

Question 3

$$(3^d - 1)N^2/K^2 = NP$$

$$K = (3^d - 1)^{1/2}(N/P)^{1/2} \simeq N^{1/2}$$

$$F(K,N) = NP + (3^d - 1)^{1/2}N^{3/2}P^{1/2} + (3^d - 1)^{1/2}N^{3/2}P^{1/2} = O(N^{3/2})$$

Question 4

- Bit interleaving
- Given box number 127

$$127=64+32+16+8+4+2+1$$

$$=2^6+2^5+2^4+2^3+2^2+2^1+2^0$$

$$127_{10}=1111111_2$$

- Given quadtree, so the dimension is 2
- Start from the end
- 1111 and 111
- So we can de-interleave as {0111,1111} and so the number of the box is {7,15}
- So the neighborhood is {6,14}{7,14}{8,14}{6,15}{7,15},{8,15}{6,16}{7,16},{8,16}
- However for 16 we require 5 bits (10000), and we are at level 4

Question 4

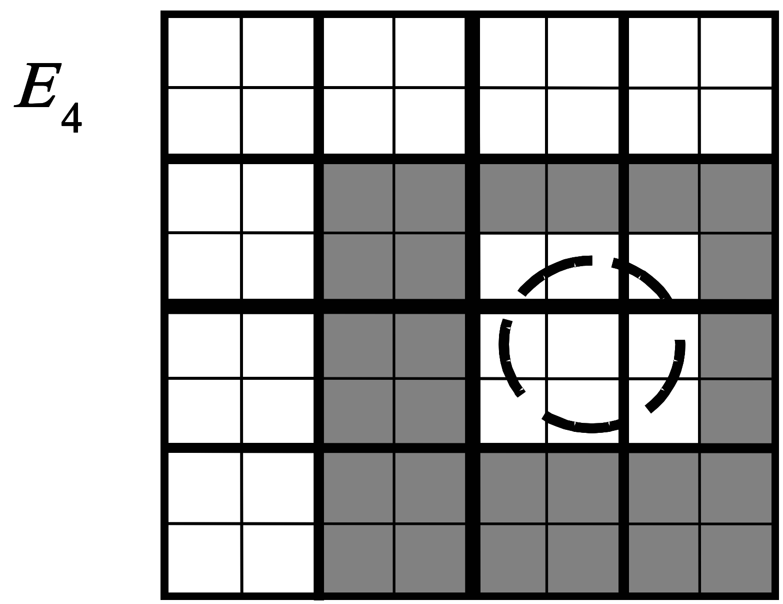
- So the neighborhood is {6,14} {7,14} {8,14} {6,15} {7,15},{8,15}
- Go back to binary
- {0110,1110} {0111,1110} {1000,1110} {0110,1111} {0111,1111} {1000,1111}
- Interleave
- {01111100} {01111110} {11010100} {01111101} {01111111} {11010101}
- 2^6+2^5+2^4+2^3+2^2=64+32+16+8+4=124
- 2^6+2^5+2^4+2^3+2^2+2^1=64+32+16+8+4+2=126
- 2^7+2^4+^2+2^0=128+64+16+4=212
- 64+32+16+8+4+1=125
- 64+32+16+8+4+2+1=127
- 128+64+16+4+1=213

DOWNWARD PASS

- Starting from level 2, build an R expansion in boxes where R expansion is valid

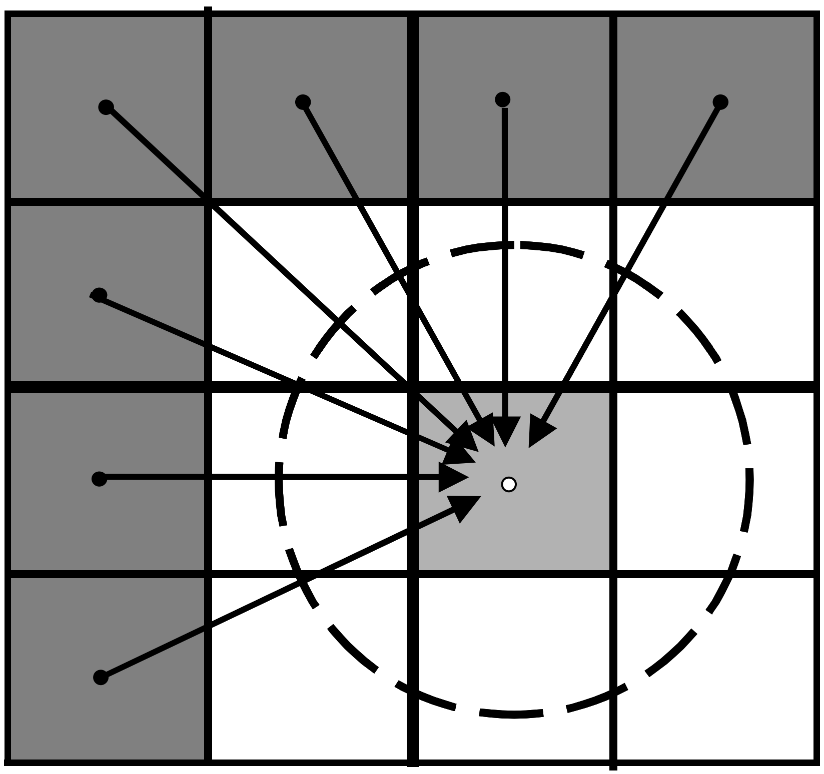
$$\Phi_4^{(n,l)}(\mathbf{y}) = \widetilde{\mathbf{D}}^{(n,l)} \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_c^{(n,l)}),$$
$$\widetilde{\mathbf{D}}^{(n,l)} = \sum_{m \in I_4(n,l)} (\mathbf{S}|\mathbf{R})\left(\mathbf{x}_c^{(n,l)} - \mathbf{x}_c^{(m,l)}\right)\mathbf{C}^{(m,l)}.$$

- Must to do $S|R$ translation
- The S expansion is not valid in boxes immediately surrounding the current box
- So we must exclude boxes in the E_4 neighborhood

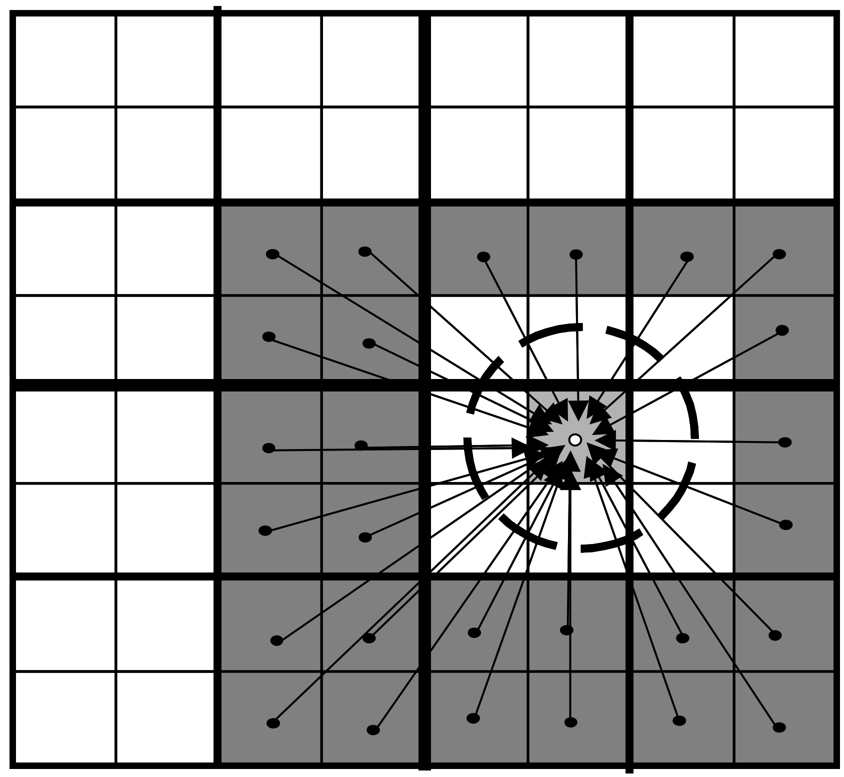


Downward Pass. Step 1.

Level 2:

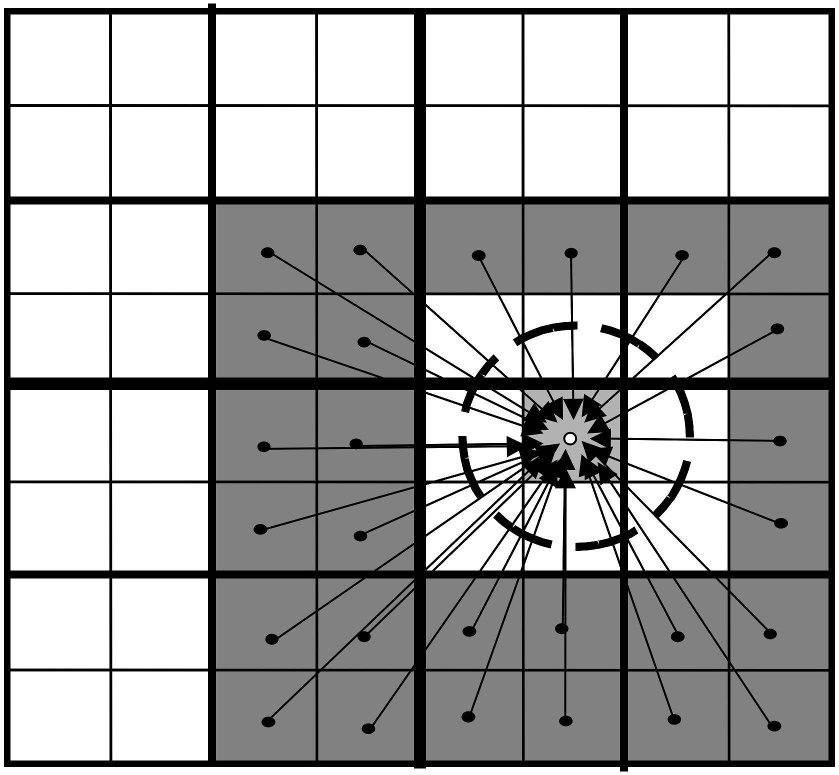


Level 3:



Downward Pass. Step 1.

THIS MIGHT BE
THE MOST EXPENSIVE
STEP OF THE ALGORITHM



Projects

- Jaynarayan Sitaraman: fast evaluation of Biot Savart velocity fields.
  - 3-D. No equation to solve …
- Ankur and Vikas: Kernel methods via FMM
  - Explore iterative methods for eigen-value problems
  - Solve some toy problems
- Changjiang Yang: Fast Gauss Transform based on Taylor series
- Ali Zandifar, Sernam Lin: Thin plate spline fitting for 2-D problems
- Azam El-Halfy, Kexue Liu: fitting of RBFs in 3-D for interpolation (graphics and vision applications

- Jun Shen – 2-D fast vortex method
- Zhihui Tang: Helmholtz equation and the FMM

Format of Project

- November 14 – due problem statement and approach of the project
  - What is the problem? Does it require matrix vector product or solution of some linear system? Prior Work?
  - What is the function \Phi? What is the S expansion? What is the R expansion? What are the S|S, S|R and R|R translation matrices?
- November 28 – due progress report
  - Direct solution of some sample problems
  - FMM solution
- December 5 – 15 minute talk and final report

Project Ideas

- Your research
- Scattered data interpolation in 1-D, 2-D and 3-D using RBFs
- ftp://ftp.stat.wisc.edu/pub/wahba/interf/rootpar2r.pdf
- The fast gauss transform and its performance in multiple dimensions
- Accelerating the BEM for the Helmholtz equation in 2D and/or 3D

- Accelerating the BEM for the Laplace equation in 2D and/or 3D
- An animation of the MLFMM ... HTML or JAVA animations of a 2-D problem (see the site on spatial data structures on Hanan Samet's site)
- Fast Fourier Transforms based on the FMM

- Eigenvalue problems using the FMM (http://www.cs.utk.edu/~dongarra/etemplates/) Using the FMM with Rayleigh
- Compressible 2-D vortex method ...
- Others (particularly would like to see problems from your research, or problems which generalize the FMM to operators which are distributive (not necessarily products)

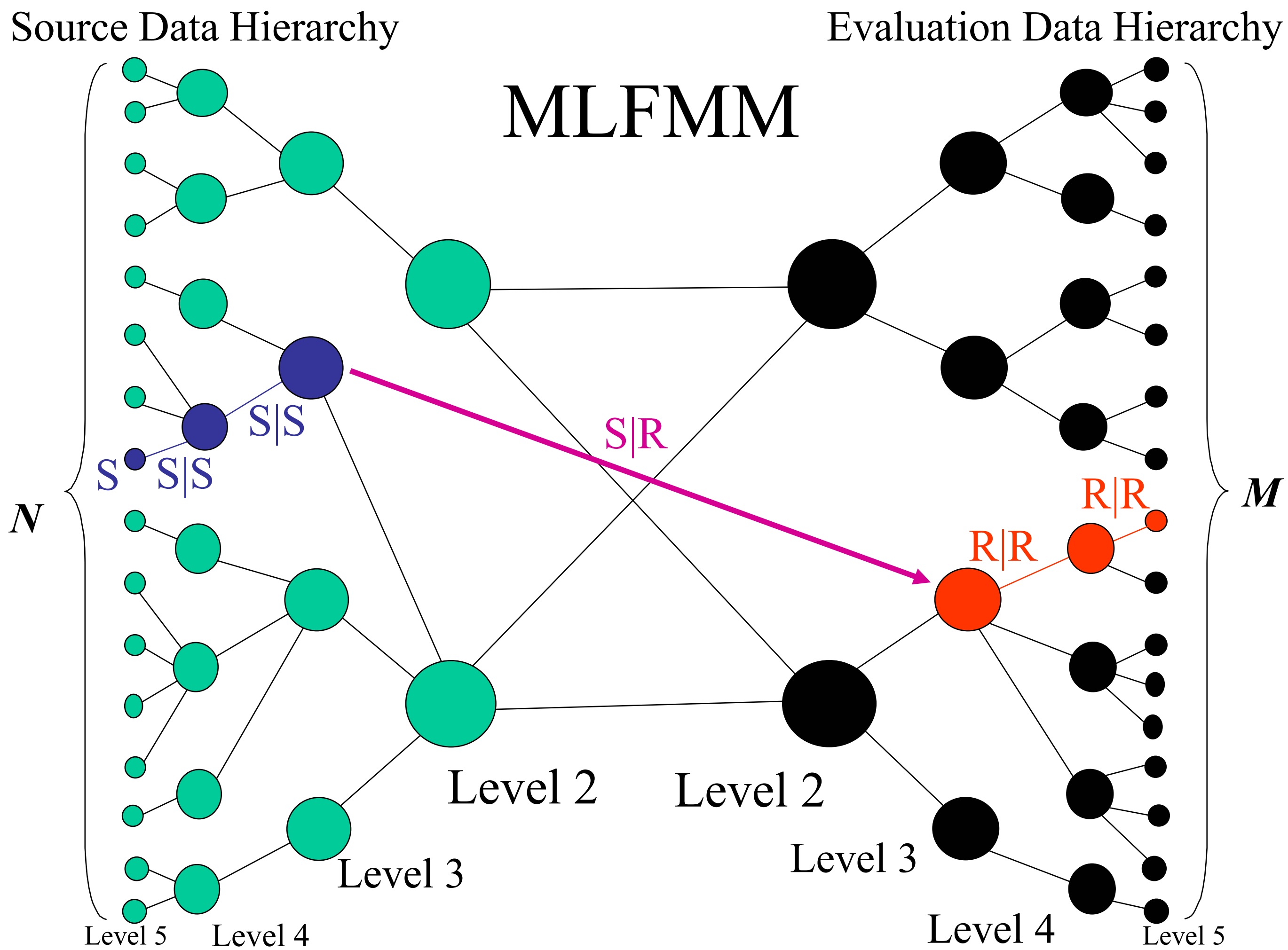
FMM CMSC 878R/AMSC 698R

Lecture 17

Outline

- Error Bounds of MLFMM
  - A scheme for error evaluation;
- Example problem
  - S-expansion error;
  - S|S-translation error;
  - S|R-translation error;
  - R|R-translation error.
- Error and Neighborhoods

A scheme for error evaluation (1)
(How one source contributes to one evaluation points)



A scheme for error evaluation (2)

$$\Phi(\mathbf{y},\mathbf{x}_k) = \sum_{m=0}^{\infty} C_m^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)})S_m(\mathbf{y}-\mathbf{x}_*^{(L)}) = \mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L)})$$

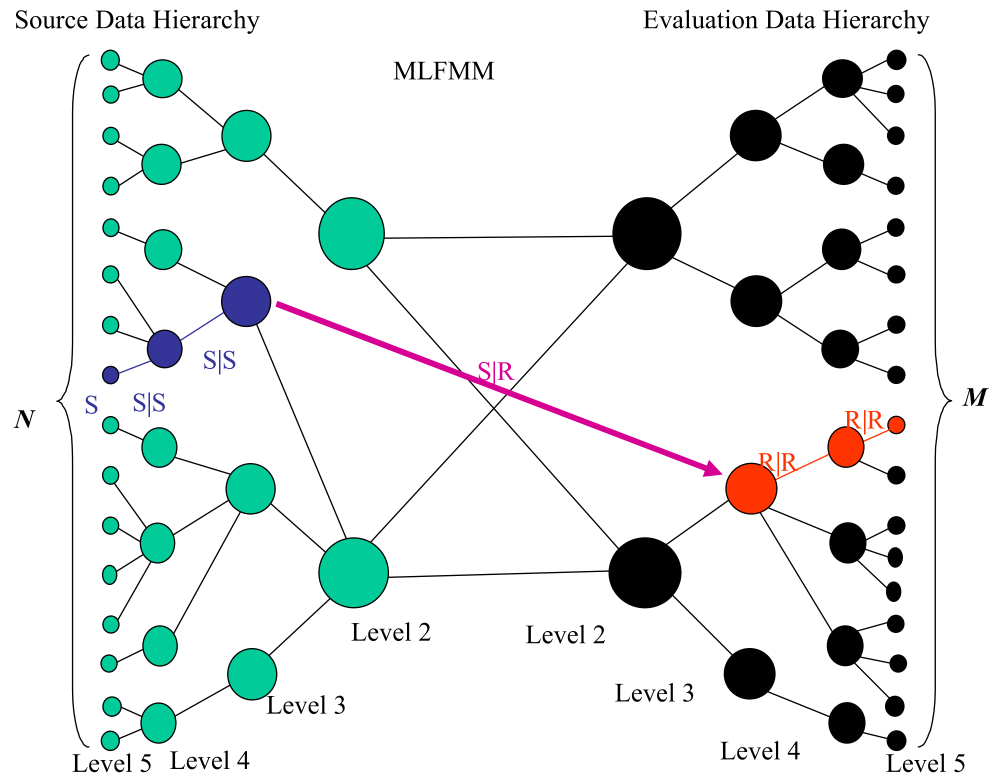
$$= \mathbf{C}^{(L-1)}(\mathbf{x}_k,\mathbf{x}_*^{(L-1)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L-1)})$$

$$= \ldots = \mathbf{C}^{(l)}(\mathbf{x}_k,\mathbf{x}_*^{(l)}) \cdot \mathbf{S}\left(\mathbf{y}-\mathbf{x}_*^{(l)}\right)$$

$$= \mathbf{D}^{(l)}(\mathbf{x}_k,\mathbf{y}_*^{(l)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l)}\right)$$

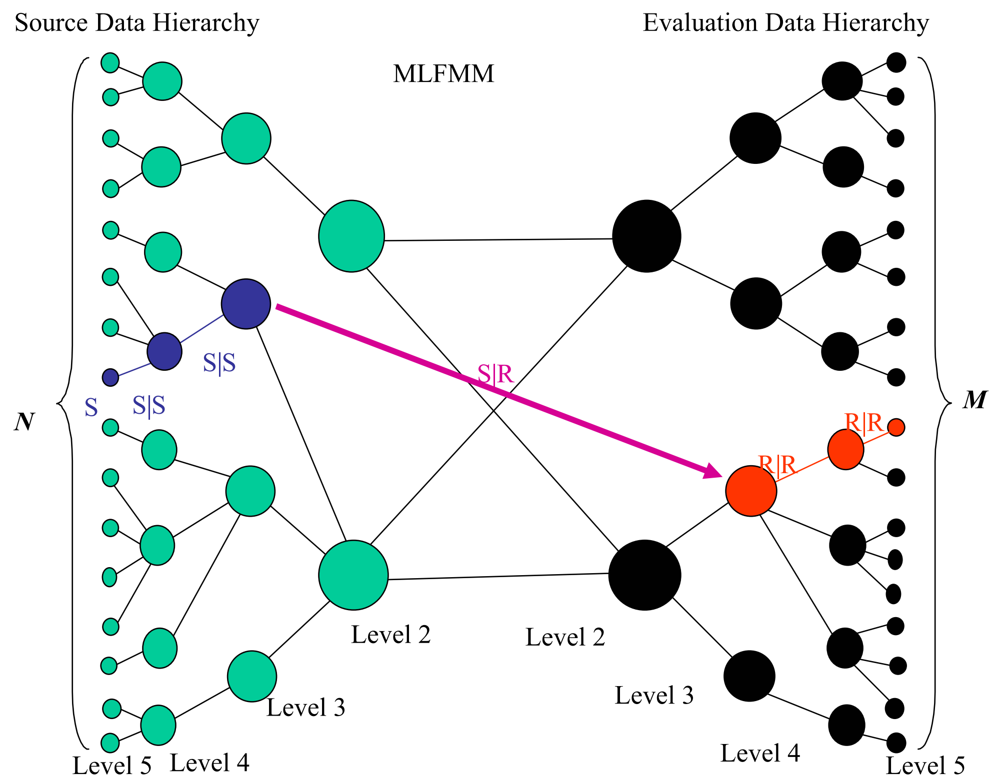
$$= \mathbf{D}^{(l+1)}(\mathbf{x}_k,\mathbf{y}_*^{(l+1)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l+1)}\right)$$

$$= \ldots = \mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) \cdot \mathbf{R}(\mathbf{y}-\mathbf{y}_*^{(L)}).$$



A scheme for error evaluation (3)

$$
\begin{aligned}
\mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) &= (\mathrm{R|R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\mathbf{D}^{(L-1)}(\mathbf{x}_k,\mathbf{y}_*^{(L-1)}) \\
&= [(\mathrm{R|R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ(\mathrm{R|R})(\mathbf{y}_*^{(L-1)}-\mathbf{y}_*^{(L-2)})]\mathbf{D}^{(L-2)}(\mathbf{x}_k,\mathbf{y}_*^{(L-2)}) \\
&= \ldots = \left[(\mathrm{R|R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ(\mathrm{R|R})(\mathbf{y}_*^{(L-1)}-\mathbf{y}_*^{(L-2)})\circ\ldots\circ(\mathrm{R|R})\left\{\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right\}\right]\mathbf{D}^{(l)}(\mathbf{x}_k,\mathbf{y}_*^{(l)}) \\
&= \left[(\mathrm{R|R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\ldots\circ(\mathrm{R|R})\left\{\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right\}\circ(\mathrm{S|R})\left\{\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right\}\right]\mathbf{C}^{(l)}(\mathbf{x}_k,\mathbf{x}_*^{(l)}) \\
&= \left[(\mathrm{R|R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\ldots\circ(\mathrm{R|R})\left\{\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right\}\circ(\mathrm{S|R})\left\{\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right\}\circ(\mathrm{S|S})\left\{\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right\}\right]\mathbf{C}^{(l+1)}(\mathbf{x}_k,\mathbf{x}_*^{(l+1)}) \\
&= \ldots \\
&= \left[(\mathrm{R|R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\ldots\circ(\mathrm{R|R})\left\{\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right\}\circ(\mathrm{S|R})\left\{\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right\}\circ(\mathrm{S|S})\left\{\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right\}\circ\ldots\circ(\mathrm{S|S})(\mathbf{x}_*^{(L-1)}-\mathbf{x}_*^{(L)})\right]\mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)})
\end{aligned}
$$



A scheme for error evaluation (4)

Consider computation of the final coefficients with p-truncated matrices

$$\begin{aligned}\mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) =& [\Pr(p)\circ(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\Pr(p)]\circ\ldots\circ\\ &\left[\Pr(p)\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\circ\Pr(p)\right]\circ\\ &\left[\Pr(p)\circ(\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right)\circ\Pr(p)\right]\circ\\ &\left[\Pr(p)\circ(\mathbf{S}|\mathbf{S})\left(\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right)\circ\Pr(p)\right]\circ\ldots\circ\\ &[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-2)}-\mathbf{x}_*^{(L-1)})\circ\Pr(p)]\circ\\ &[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-1)}-\mathbf{x}_*^{(L)})\circ\Pr(p)]\mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)})\end{aligned}$$

These truncation operators can be skipped! ($\Pr^2=\Pr$)

So:

$$\begin{aligned}\mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) =& [\Pr(p)\circ(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})]\circ\ldots\circ\\ &\left[\Pr(p)\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\right]\circ\\ &\left[\Pr(p)\circ(\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right)\right]\circ\\ &\left[\Pr(p)\circ(\mathbf{S}|\mathbf{S})\left(\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right)\right]\circ\ldots\circ\\ &[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-2)}-\mathbf{x}_*^{(L-1)})]\circ\\ &[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-1)}-\mathbf{x}_*^{(L)})]\circ\Pr(p)\mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)})\end{aligned}$$

A scheme for error evaluation (5)

p-truncated functions:

$$\widehat{\Phi}_L(\mathbf{y},\mathbf{x}_k) = \widehat{\mathbf{C}}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L)})$$

$$\widehat{\Phi}_{L-1}(\mathbf{y},\mathbf{x}_k) = \widehat{\mathbf{C}}^{(L-1)}(\mathbf{x}_k,\mathbf{x}_*^{(L-1)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L-1)})$$

$$\ldots$$

$$\widehat{\Phi}_l(\mathbf{y},\mathbf{x}_k) = \widehat{\mathbf{C}}^{(l)}(\mathbf{x}_k,\mathbf{x}_*^{(l)}) \cdot \mathbf{S}\left(\mathbf{y}-\mathbf{x}_*^{(l)}\right)$$

$$\widehat{\Psi}_l(\mathbf{y},\mathbf{x}_k) = \widehat{\mathbf{D}}^{(l)}(\mathbf{x}_k,\mathbf{y}_*^{(l)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l)}\right)$$

$$\widehat{\Psi}_{l+1}(\mathbf{y},\mathbf{x}_k) = \widehat{\mathbf{D}}^{(l+1)}(\mathbf{x}_k,\mathbf{y}_*^{(l+1)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l+1)}\right)$$

$$\ldots$$

$$\widehat{\Psi}_L(\mathbf{y},\mathbf{x}_k) = \widehat{\mathbf{D}}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) \cdot \mathbf{R}(\mathbf{y}-\mathbf{y}_*^{(L)}).$$

The error comes only from truncation operator

$$\widehat{\mathbf{C}}^{(\alpha)} = \Pr(p) \circ (\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)})\widehat{\mathbf{C}}^{(\alpha+1)}, \quad \alpha = L-1,\ldots,l$$

$$\widehat{\mathbf{D}}^{(l)} = \Pr(p) \circ (\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)} - \mathbf{x}_*^{(l)}\right)\widehat{\mathbf{C}}^{(l)},$$

$$\widehat{\mathbf{D}}^{(\alpha)} = \Pr(p) \circ (\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(\alpha)} - \mathbf{y}_*^{(\alpha+1)})\widehat{\mathbf{C}}^{(\alpha+1)}, \quad \alpha = l,\ldots,L-1$$

Truncated Translation Theorem (see Lecture 11)

Let $\{F_n(\mathbf{y})\}$ and $\{G_n(\mathbf{y})\}$ be two expansion bases in Ω, and the reexpansion series converges everywhere in Ω :

$$\forall \mathbf{y} \in \Omega, \quad F_n(\mathbf{y}) = \sum_{m=0}^{\infty} (F|G)_{mn} G_m(\mathbf{y}), \quad n = 0, 1, 2, \ldots$$

Let also $\{A_n\}$ be a set of coefficients, such that the duble sum converges absolutely and uniformly in Ω :

$$\forall \mathbf{y} \in \Omega, \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_n (F|G)_{mn} G_m(\mathbf{y}) = \Phi(\mathbf{y}),$$

$$\forall \epsilon, \exists p(\epsilon), \quad \sum_{n=0}^{\infty} \sum_{m=p}^{\infty} |A_n (F|G)_{mn} G_m(\mathbf{y})| < \epsilon, \quad \sum_{n=p}^{\infty} \sum_{m=0}^{\infty} |A_n (F|G)_{mn} G_m(\mathbf{y})| < \epsilon.$$

Then

$$\left| \Phi(\mathbf{y}) - \sum_{n=0}^{p-1} \sum_{m=0}^{p-1} A_n (F|G)_{mn} G_m(\mathbf{y}) \right| < 2\epsilon.$$

A scheme for error evaluation (5)

For uniformly and absolutely convergent series:

$$
\begin{aligned}
&\left|\widehat{\Phi}_{\alpha}(\mathbf{y},\mathbf{x}_k) - \widehat{\Phi}_{\alpha+1}(\mathbf{y},\mathbf{x}_k)\right| \\
&= \left|\sum_{m=0}^{p-1} \widehat{C_m}^{(\alpha)} S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)}) - \sum_{n=0}^{p-1} \widehat{C_n}^{(\alpha+1)} S_n(\mathbf{y}-\mathbf{x}_*^{(\alpha+1)})\right| \\
&= \left|\sum_{m=0}^{p-1}\sum_{n=0}^{p-1} (S|S)_{mn}(\mathbf{x}_*^{(\alpha)}-\mathbf{x}_*^{(\alpha+1)})\widehat{C_n}^{(\alpha+1)} S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right. \\
&\quad \left. - \sum_{n=0}^{p-1} \widehat{C_n}^{(\alpha+1)} \sum_{m=0}^{\infty} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)}-\mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| \\
&= \left|\sum_{n=0}^{p-1} \widehat{C_n}^{(\alpha+1)} \sum_{m=p}^{\infty} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)}-\mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| \\
&\leqslant \sum_{n=0}^{p-1}\sum_{m=p}^{\infty} \left|\widehat{C_n}^{(\alpha+1)} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)}-\mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| \\
&< \sum_{n=0}^{\infty}\sum_{m=p}^{\infty} \left|\widehat{C_n}^{(\alpha+1)} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)}-\mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| < \epsilon_{\max}(p).
\end{aligned}
$$

A scheme for error evaluation (6)

For uniformly and absolutely convergent series it is possible to find such $\varepsilon_{\max}(p)$ that for given minimum(maximum) translation distance the max abs difference between two subsequent functions is smaller than $\varepsilon_{\max}(p)$.

In this case the total error of FMM does not exceed:

$$FMMError \leqslant N\left[\epsilon_{\max}^{(\exp)}(p) + (L-2)\epsilon_{\max}^{(S|S)}(p) + \epsilon_{\max}^{(S|R)}(p) + (L-2)\epsilon_{\max}^{(R|R)}(p)\right].$$

$$\lim_{p\to\infty}\epsilon_{\max}^{(\exp)}(p) = 0, \quad \lim_{p\to\infty}\epsilon_{\max}^{(S|S)}(p) = 0, \quad \lim_{p\to\infty}\epsilon_{\max}^{(S|R)}(p) = 0, \quad \lim_{p\to\infty}\epsilon_{\max}^{(R|R)}(p) = 0.$$

If

$$\epsilon(p) = \max\left(\epsilon_{\max}^{(\exp)}(p), \epsilon_{\max}^{(S|S)}(p), \epsilon_{\max}^{(S|R)}(p), \epsilon_{\max}^{(R|R)}(p)\right),$$

$$FMMError \leqslant 2N(L-1)\epsilon(p).$$

Error of exact $R|R$, $S|S$, and $S|R$-translation (Lecture 8…)

If

$$\|\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})\| < \epsilon,$$

then

$$\|(\mathcal{R}|\mathcal{R})(\mathbf{t})(\Phi(\mathbf{y}) - \Phi^p(\mathbf{y}))\| = \|(\mathcal{R}|\mathcal{R})(\mathbf{t})\|\,\|\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})\| < \epsilon,$$

$$\|(\mathcal{S}|\mathcal{S})(\mathbf{t})(\Phi(\mathbf{y}) - \Phi^p(\mathbf{y}))\| = \|(\mathcal{S}|\mathcal{S})(\mathbf{t})\|\,\|\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})\| < \epsilon,$$

$$\|(\mathcal{S}|\mathcal{R})(\mathbf{t})(\Phi(\mathbf{y}) - \Phi^p(\mathbf{y}))\| = \|(\mathcal{S}|\mathcal{R})(\mathbf{t})\|\,\|\Phi(\mathbf{y}) - \Phi^p(\mathbf{y})\| < \epsilon.$$

Example Problem

Problem:
Evaluate the MLFMM error for computation of function`

$$v(y) = \sum_{k=0}^{N} u_k \Phi(y, x_k),$$

$$\Phi(y, x_k) = \frac{1}{y - x_k},$$

where y and x_k are points in a box of size D and space is subdivided by the binary tree to the maximum level L.

From Lecture 6…

$|y - x_*| < |x_i - x_*|$:

R-expansion

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y - x_*),$$

$$a_m(x_i, x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, \ldots,$$

$$R_m(y - x_*) = (y - x_*)^m, \quad m = 0, 1, \ldots$$

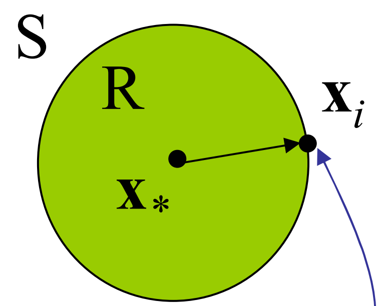
$|y - x_*| > |x_i - x_*|$:

S-expansion

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*),$$

$$b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \ldots,$$

$$S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \ldots$$



Singular Point is located at the Boundary of regions for the R- and S-expansions!

From Lecture 8…

$(|y - x_*| < |t|)$

$$S_n(y - x_* + t) = (t + y)^{-n-1} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^n} (y - x_*)^m$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m S_n(t)}{dt^n} R_m(y - x_*) = \sum_{m=0}^{\infty} (S|R)_{mn}(t) R_m(y - x_*).$$

So

$$(S|R)_{mn}(t) = \frac{1}{m!} \frac{d^m S_n(t)}{dt^n} = \frac{(-1)^m (m+n)!}{m! n! t^{n+m+1}}.$$

$$(\mathbf{S}|\mathbf{R})(t) = \begin{pmatrix} t^{-1} & t^{-2} & t^{-3} & \ldots \\ -t^{-2} & -2t^{-3} & -3t^{-4} & \ldots \\ t^{-3} & 3t^{-4} & 6t^{-5} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$