FMM CMSC 878R/AMSC 698R

Lecture 12

Outline

- MLFMM Algorithm (Continuation)
  - Downward Pass;
  - Final Summation;
- Asymptotic Complexity of the MLFMM;
  - Setting Data Structure;
  - S-expansion;
  - Upward Pass;
  - Downward Pass;
  - Final Summation;
  - Total Complexity.

Downward Pass. Step 1.

Step 1. Steps 1 and 2 should be performed recursively for levels $l = 2, \ldots L$ of space subdivision. At this step form coefficients of regular expansion for function $\Phi_4^{(n,l)}(\mathbf{y})$. To build local expansion near the center of each box at level l coefficients $\mathbf{C}^{(m,l)}, m \in I_4(n,l)$ should be $(\mathbf{S}|\mathbf{R})$- translated to the center of this box. So we have

$$\Phi_4^{(n,l)}(\mathbf{y}) = \widetilde{\mathbf{D}}^{(n,l)} \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_c^{(n,l)}),$$

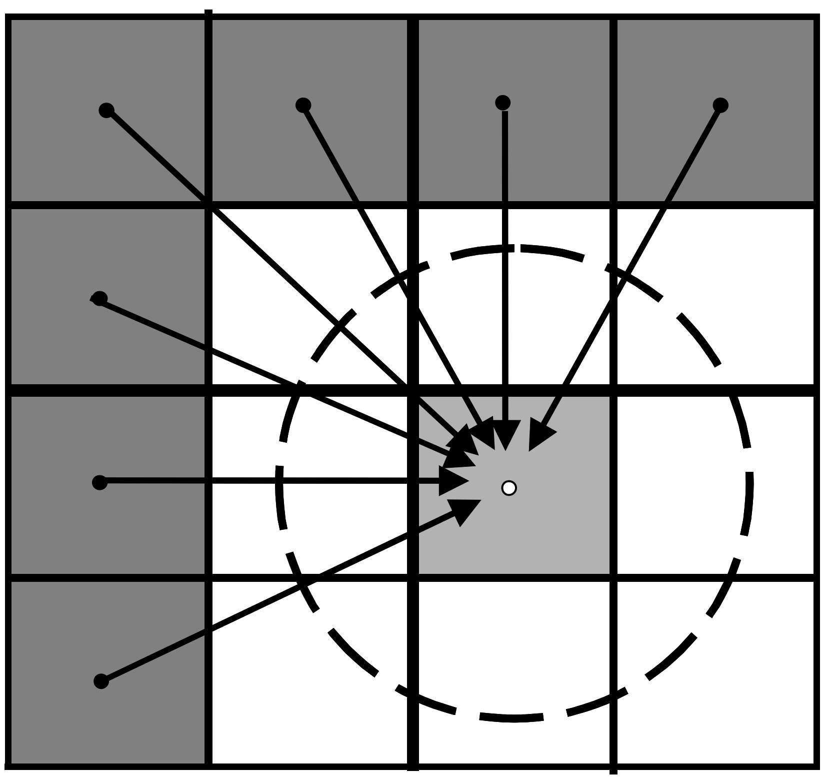
$$\widetilde{\mathbf{D}}^{(n,l)} = \sum_{m \in I_4(n,l)} (\mathbf{S}|\mathbf{R})\left(\mathbf{x}_c^{(n,l)} - \mathbf{x}_c^{(m,l)}\right)\mathbf{C}^{(m,l)}.$$

Since each box of level l is separated from boxes of $I_4(n,l)$ by a sphere drown near its center, then the far-to-local translation is applicable. Note that summation over empty boxes $m \in I_4(n,l)$ of set X can be skipped.

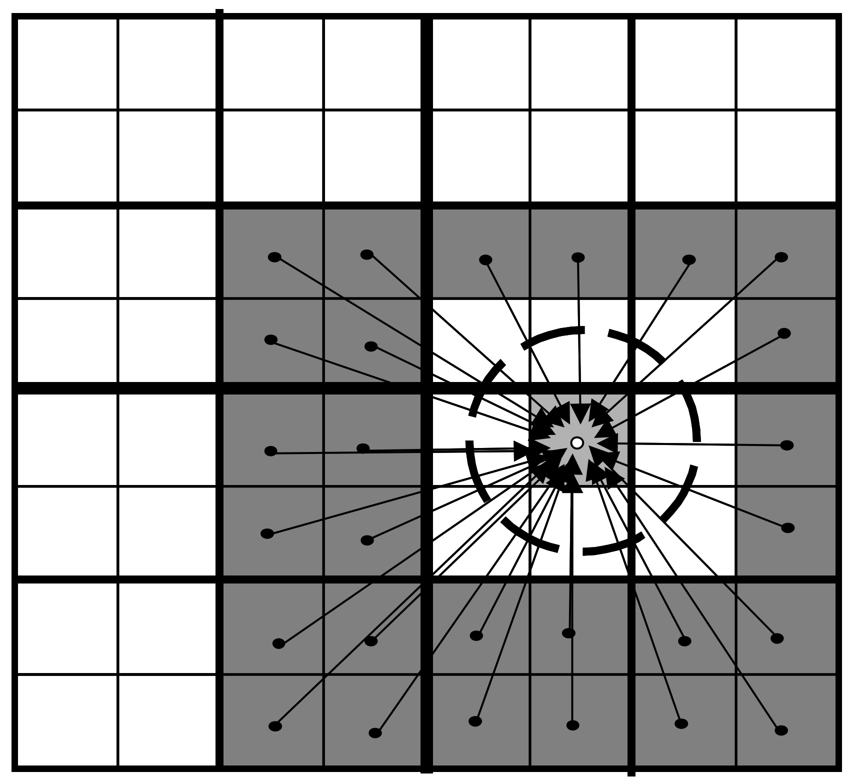
Note that this is conversion from the Source Hierarchy to Evaluation Hierarchy!

Downward Pass. Step 1.

Level 2:

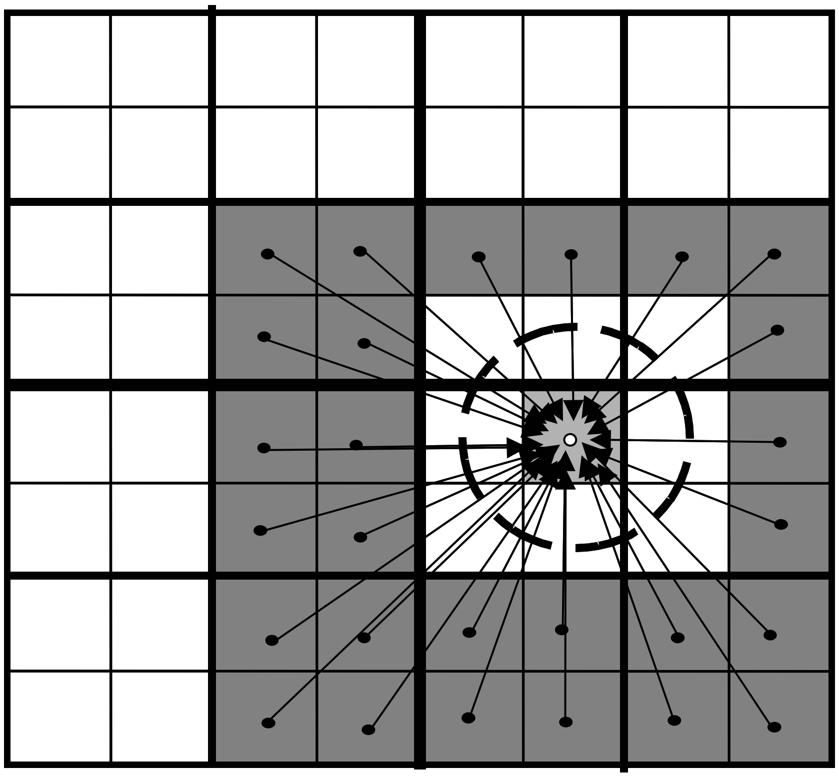


Level 3:



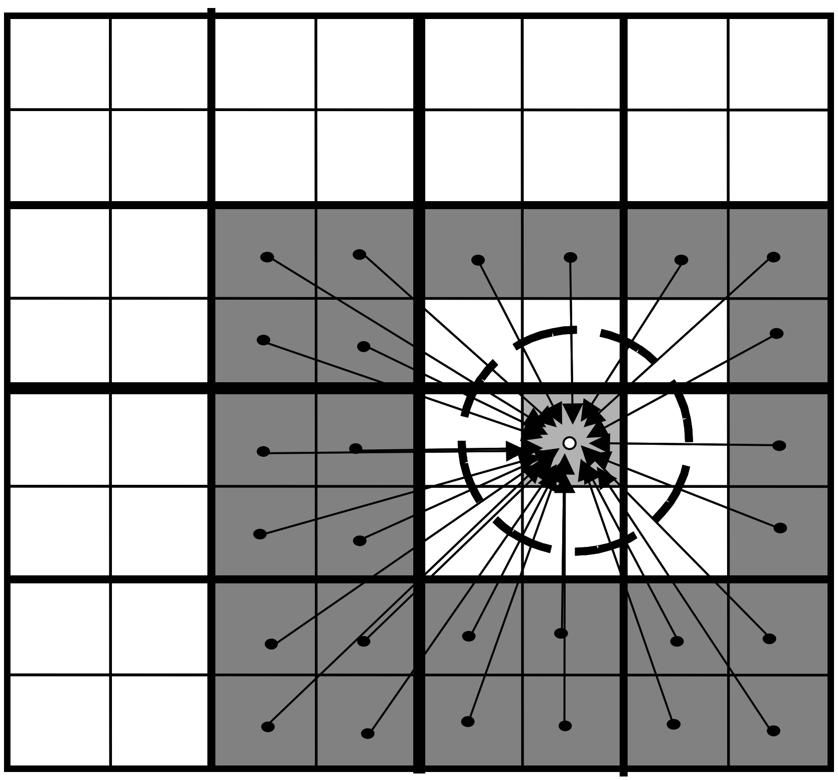
Downward Pass. Step 1.

THIS MIGHT BE
THE MOST EXPENSIVE
STEP OF THE ALGORITHM



Downward Pass. Step 1.

$$P_4 = PowerOfE_4Neighborhood = 3^d 2^d - 3^d = 3^d\left(2^d - 1\right)$$



$d = 1: \quad P_4 = 3,$

$d = 2: \quad P_4 = 27,$

$d = 3: \quad P_4 = 189,$

$d = 4: \quad P_4 = 1215,$

...

Exponential Growth

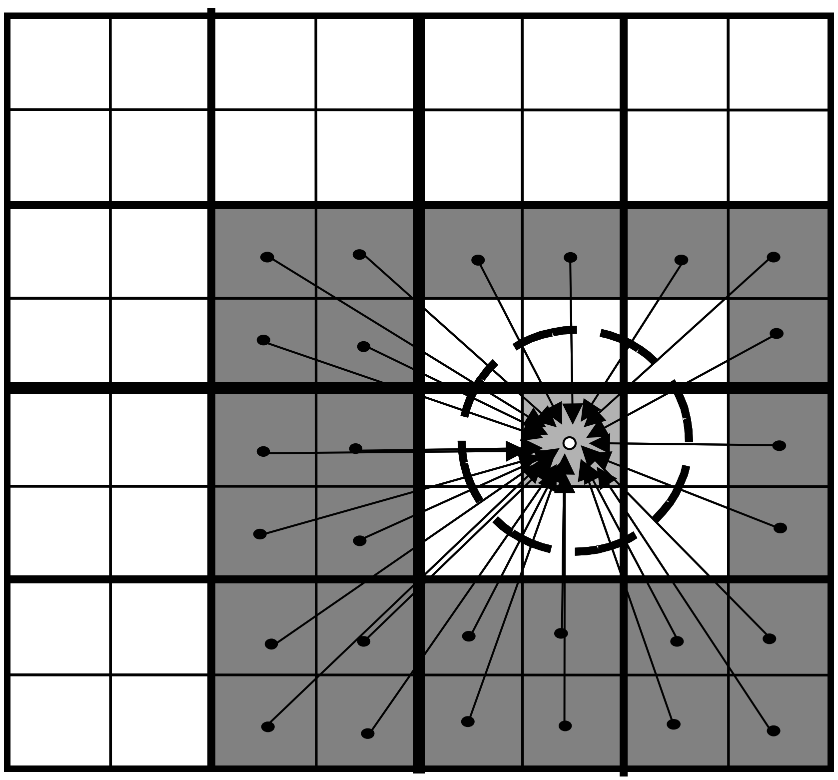
Total number of S|R-translations per 1 box in *d*-dimensional space

(far from the domain boundaries)

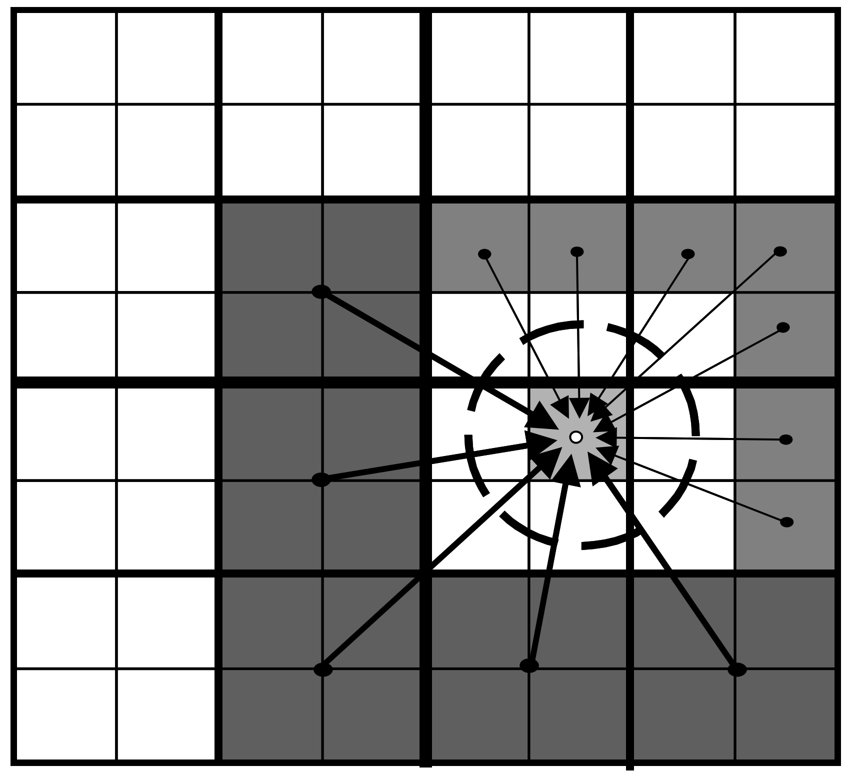
It is worth to think about optimizations

Downward Pass. Step 1.
Operation Reduction Trick.

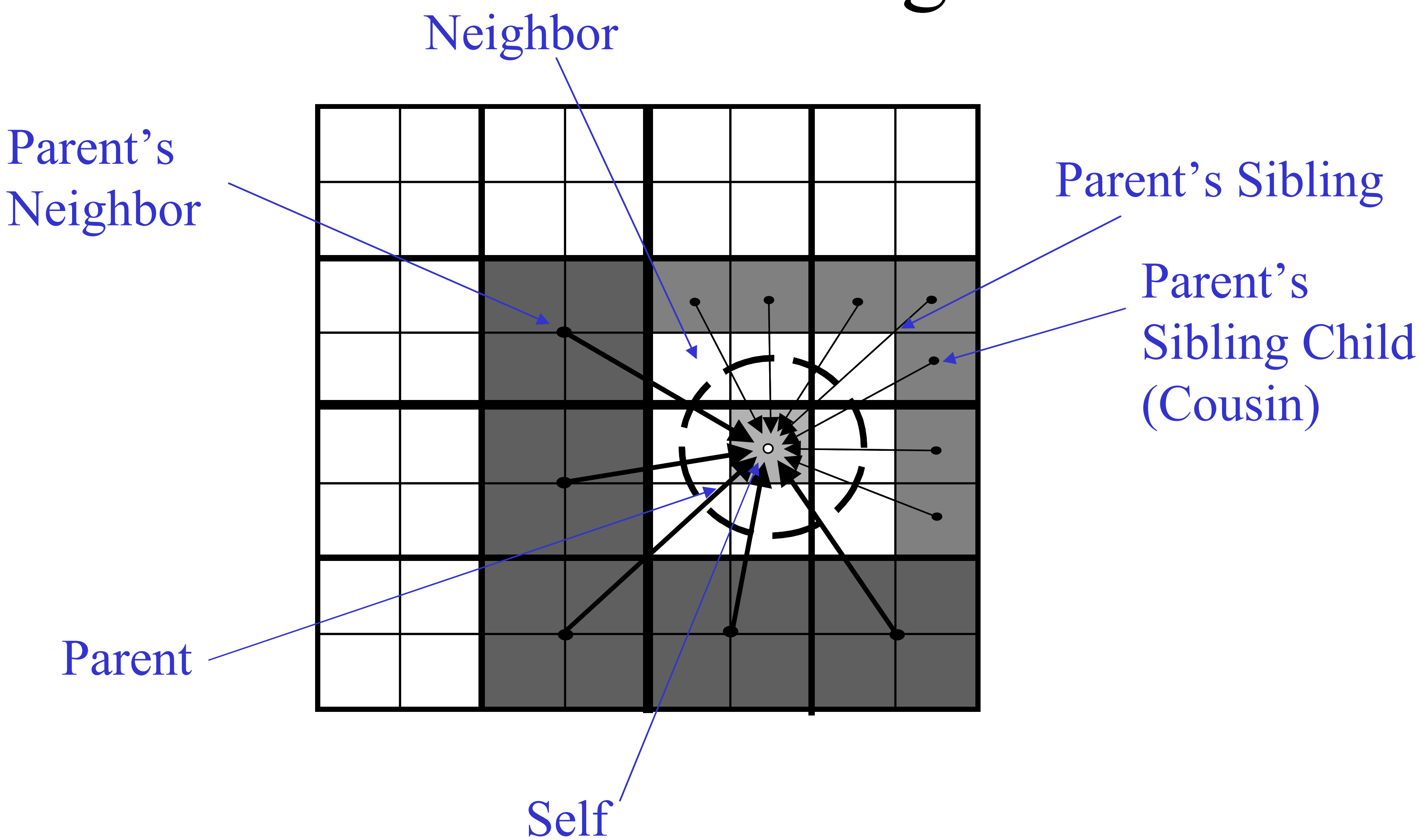
Original



Reduced



Downward Pass. Step 1. Relatives and Neighbors



Downward Pass. Step 1. Reduced Number of Translations

Parent's Neighborhood — Neighborhood — Parent+Its Siblings — Their Children — Self+Siblings+Cousins

$$P_4^{(reduced)} = PowerOfE_4Neighborhhood = 3^d - 2^d + 2^d 2^d - 3^d = 2^d\left(2^d - 1\right)$$

Savings:

$$P_4 - P_4^{(reduced)} = 3^d\left(2^d - 1\right) - 2^d\left(2^d - 1\right) = \left(3^d - 2^d\right)\left(2^d - 1\right),$$

$$P_4 / P_4^{(reduced)} = 3^d\left(2^d - 1\right)/2^d\left(2^d - 1\right) = \left(\frac{3}{2}\right)^d.$$

What this really means:

d	P_4	$P_4^{(reduced)}$	$P_4 - P_4^{(reduced)}$	$P_4/P_4^{(reduced)}$
1	3	2	1	1.5
2	27	12	15	2.25
3	189	56	133	3.375
4	1215	240	975	5.0625
…	…	…	…	…

Downward Pass. Step 1. (With Reduced Translations)

$$I_4(n,l) = I_4^1(n,l) \cup I_4^2(n,l),$$
$$Parent\left(I_4^1(n,l)\right) \cap Parent\{Neighbor(n,l)\} \neq \emptyset,$$
$$Parent\left(I_4^2(n,l)\right) \cap Parent\{Neighbor(n,l)\} = \emptyset.$$

where $I_4^1(n,l)$ is the set of boxes of level l which parent boxes include boxes from the set $\{Neighbor(n,l)\}$ and $I_4^2(n,l)$ is the set of boxes which parents do not include the neighbor boxes of (n,l). Thus all boxes of the set $I_4^2(n,l)$ can be grouped according their parents and the set $Parent\left(I_4^2(n,l)\right)$ is a set of boxes of the coarser level $l-1$ which is located in the domain $E_4(n,l)$, but is separated from the box (n,l). Therefore instead we can write

$$\Phi_4^{(n,l)}(\mathbf{y}) = \widetilde{\mathbf{D}}^{(n,l)} \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_c^{(n,l)}),$$
$$\widetilde{\mathbf{D}}^{(n,l)} = \sum_{m \in I_4^1(n,l)} (\mathbf{S}|\mathbf{R})\left(\mathbf{x}_c^{(n,l)} - \mathbf{x}_c^{(m,l)}\right)\mathbf{C}^{(m,l)} + \sum_{m \in Parent\left(I_4^2(n,l)\right)} (\mathbf{S}|\mathbf{R})\left(\mathbf{x}_c^{(n,l)} - \mathbf{x}_c^{(m,l-1)}\right)\mathbf{C}^{(m,l-1)}.$$

Boxes of the Same Level

Boxes of the Parent Level

Note that for level $l = 2$ the latter sum in the equation for $\widetilde{\mathbf{D}}^{(n,l)}$ should be set to zero, since the set $Parent\left(I_4^2(n,2)\right)$ is empty.

Downward Pass. Step 2.

Step 2. At $l = 2$ we have

$$\Phi_3^{(n,2)}(\mathbf{y}) = \Phi_4^{(n,2)}(\mathbf{y}), \quad \mathbf{D}^{(n,2)} = \widetilde{\mathbf{D}}^{(n,2)},$$

Form $\Phi_3^{(n,l)}(\mathbf{y})$ (or expansion coefficients of this function) by adding $\Phi_4^{(Parent(n),l-1)}(\mathbf{y})$ to $(\mathbf{R}|\mathbf{R})$- translated coefficients of the parent box to the child center:

$$\Phi_3^{(n,l)}(\mathbf{y}) = \mathbf{D}^{(n,l)} \circ \mathbf{R}(\mathbf{y} - \mathbf{x}_c^{(n,l)}),$$

$$\mathbf{D}^{(n,l)} = \widetilde{\mathbf{D}}^{(n,l)} + (\mathbf{R}|\mathbf{R})\left(\mathbf{x}_c^{(n,l)} - \mathbf{x}_c^{(m,l-1)}\right)\mathbf{D}^{(m,l-1)}, \quad m = Parent(n).$$

Downward Pass. Step 2.

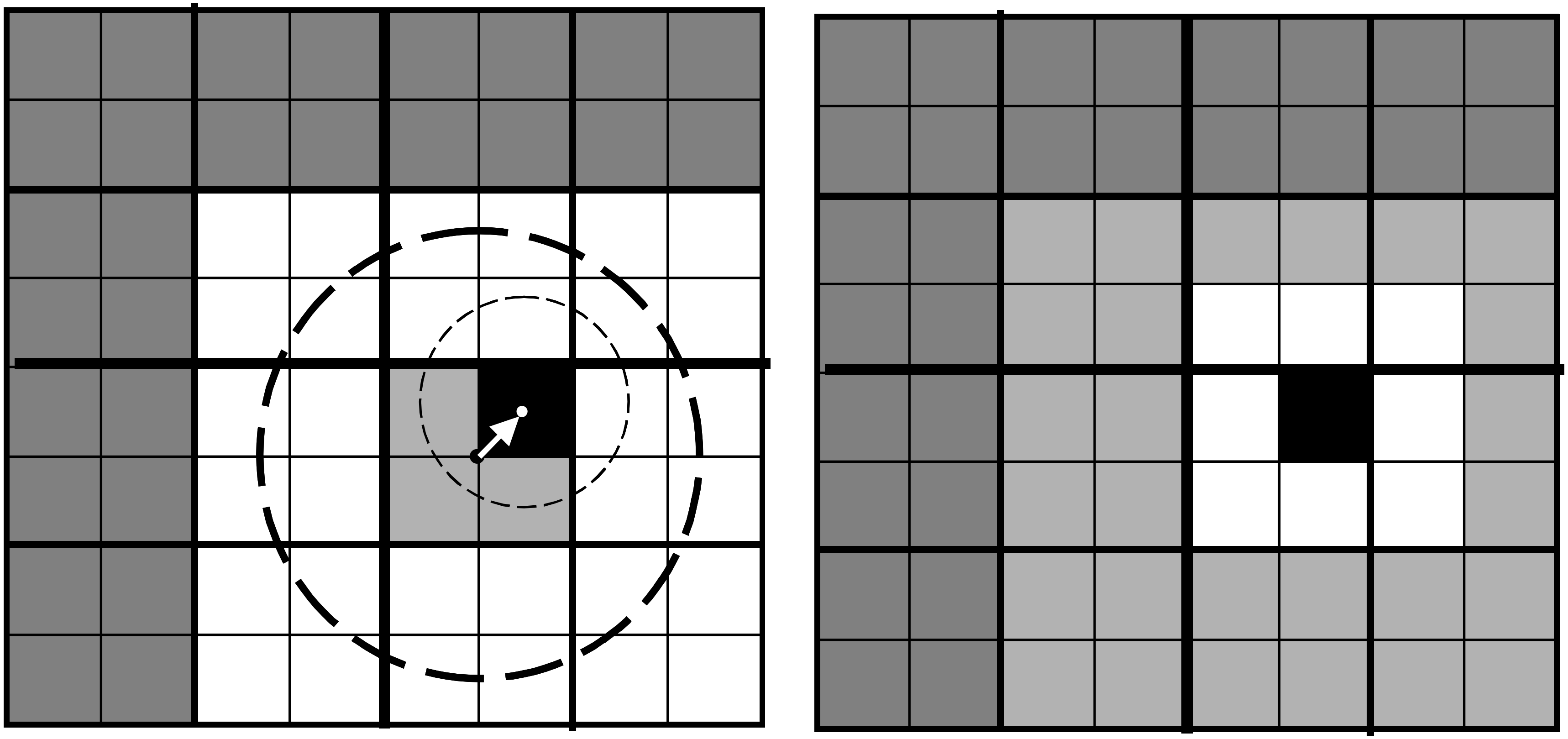


Figure shows that local-to-local translation is applicable in this case (smaller sphere is located completely inside the larger sphere), and junction of structures $E_3(n,l)$ and $E_4(n,l+1)$ produces $E_3(n,l+1)$:

$$E_3(n,l+1) = E_3(n,l) \cup E_4(n,l+1).$$

Result of the Downward Pass

In the entire hierarchy of boxes containing *evaluation points* R-expansion coefficients for potentials due to *sources* outside each *evaluation point* neighborhood (domains E_3) are found. Expansions are valid in E_1 domains.

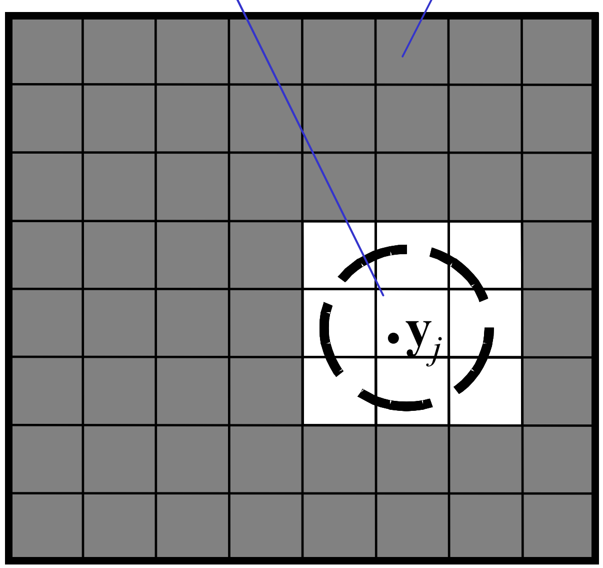
Final Summation

As soon as coefficients $\mathbf{D}^{(n,L)}$ are determined total potential can be computed for any point $\mathbf{y}_j \in E_1(0,0)$, where $\Phi_2^{(n,l)}(\mathbf{y})$ can be computed straightforward. So:

$$v_j = \Phi(\mathbf{y}_j) = \sum_{\mathbf{x}_i \in E_2(n,L)} u_i \Phi(\mathbf{y}_j, \mathbf{x}_i) + \mathbf{D}^{(n,L)} \circ \mathbf{R}(\mathbf{y}_j - \mathbf{x}_c^{(n,L)}), \quad \mathbf{y}_j \in E_1(n,L).$$

Contribution of E_2

Contribution of E_3



Complexity of the MLFMM

Let's do some evaluations for regular mesh in *d*-dimensional space and *M*=*N*:

$$N = 2^{L_* d}$$

In this case max level of space subdivision for clustering (grouping) parameter s is:

$$L = L_{\max} = \left[\frac{1}{d}\log_2 N - \frac{1}{d}\log_2 s\right] = L_* - \left[\frac{1}{d}\log_2 s\right].$$

Setting Data Structure

Basic Procedures

- Scale source and evaluation data to have the computational domain of size of a unit box.
- Sort data using interleaving technique.
- Determine the level of space subdivision with 2^d-tree to have s sources at the finest subdivision level, L_{max}.

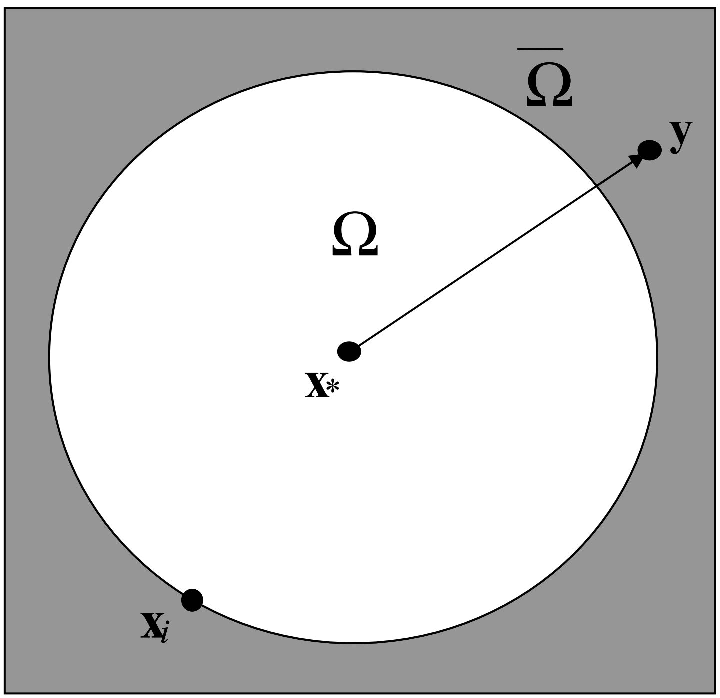
Comes from here

Total Cost

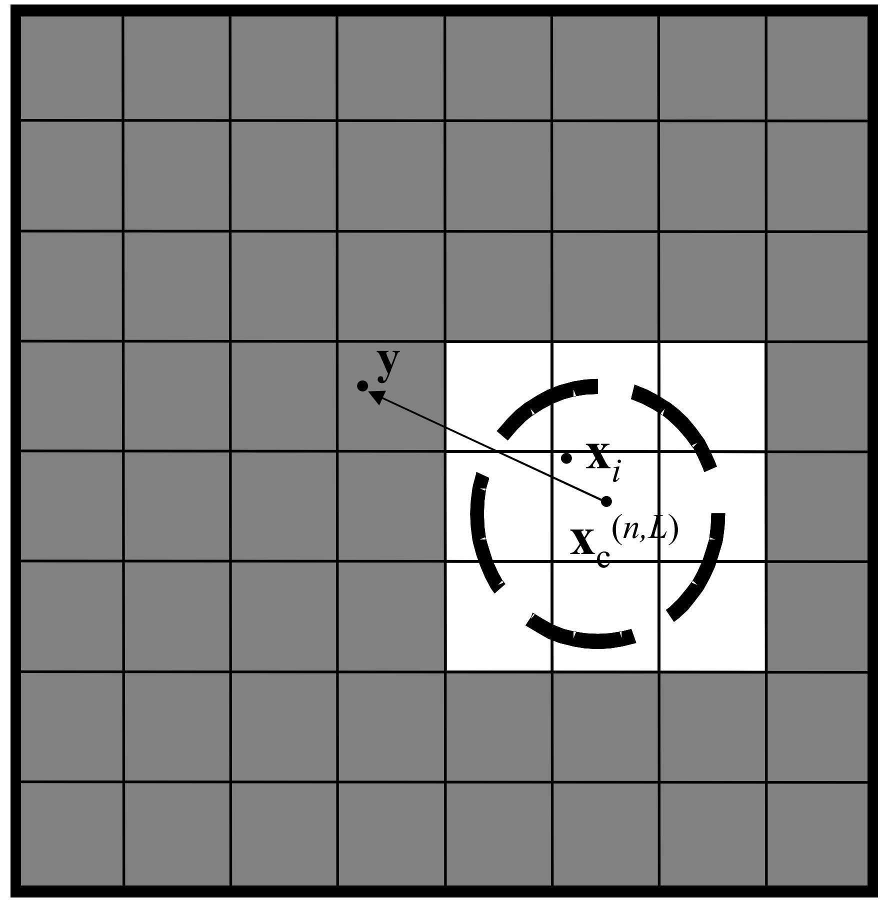
$$CostSetting = O(N \log N)$$

Upward Pass. Step 1.

S-expansion valid in $\overline{\Omega}$



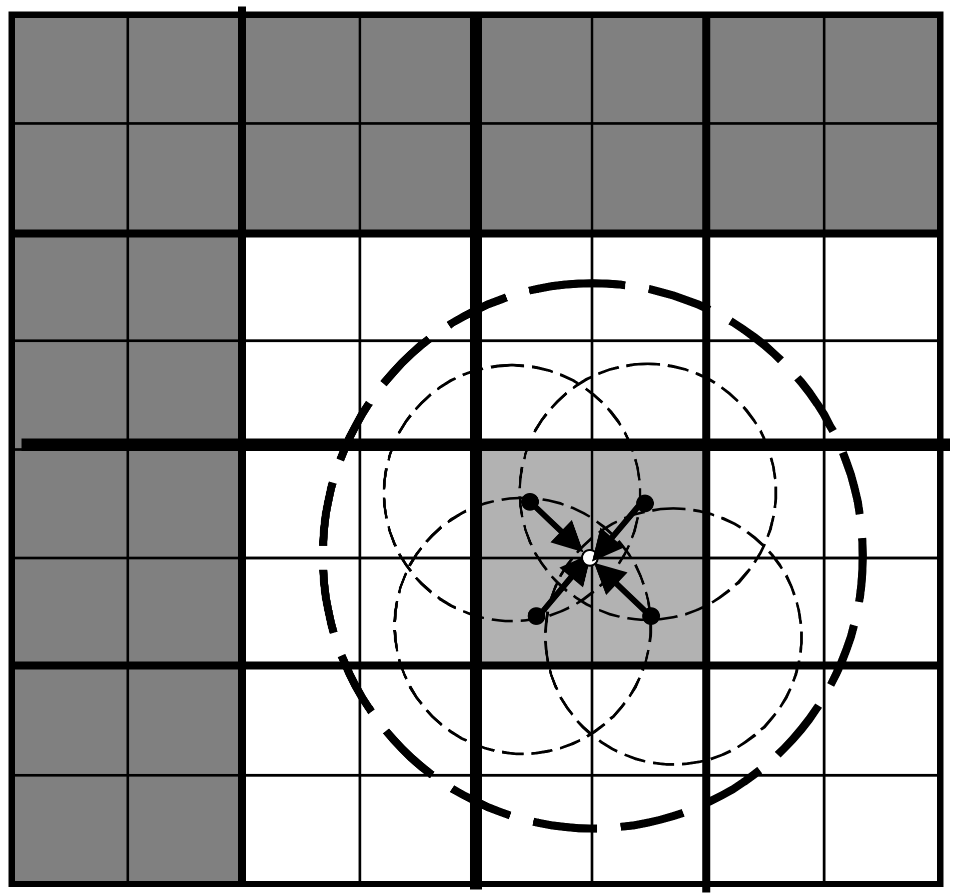
E_3



S-expansion valid in $E_3(n,L)$

Upward Pass. Step 2.

S|S-translation.
Build potential for the parent box (find its S-expansion).



Downward Pass. Step 1.
Operation Reduction Trick.

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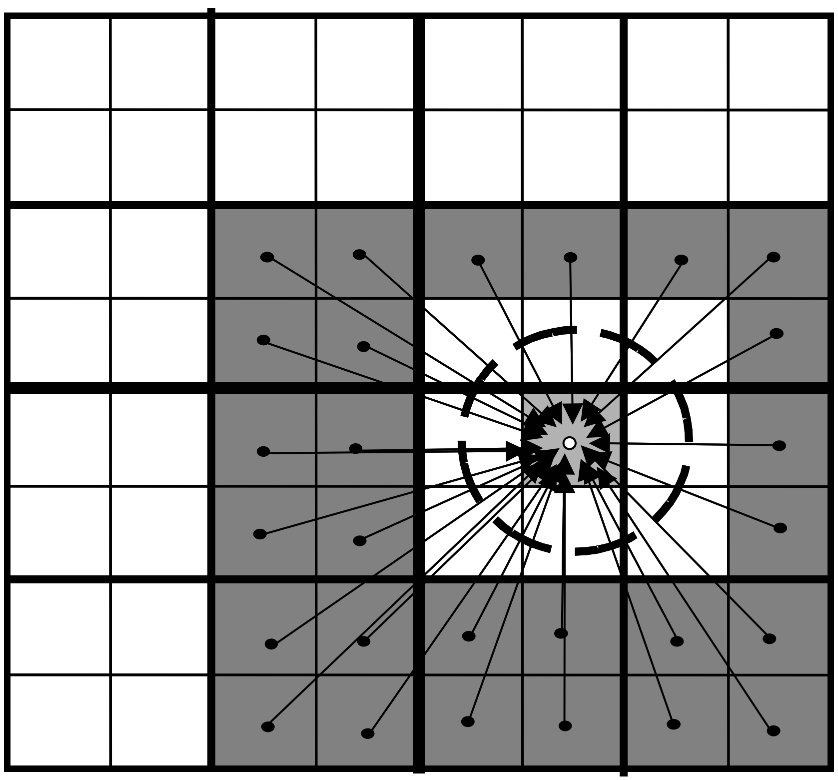
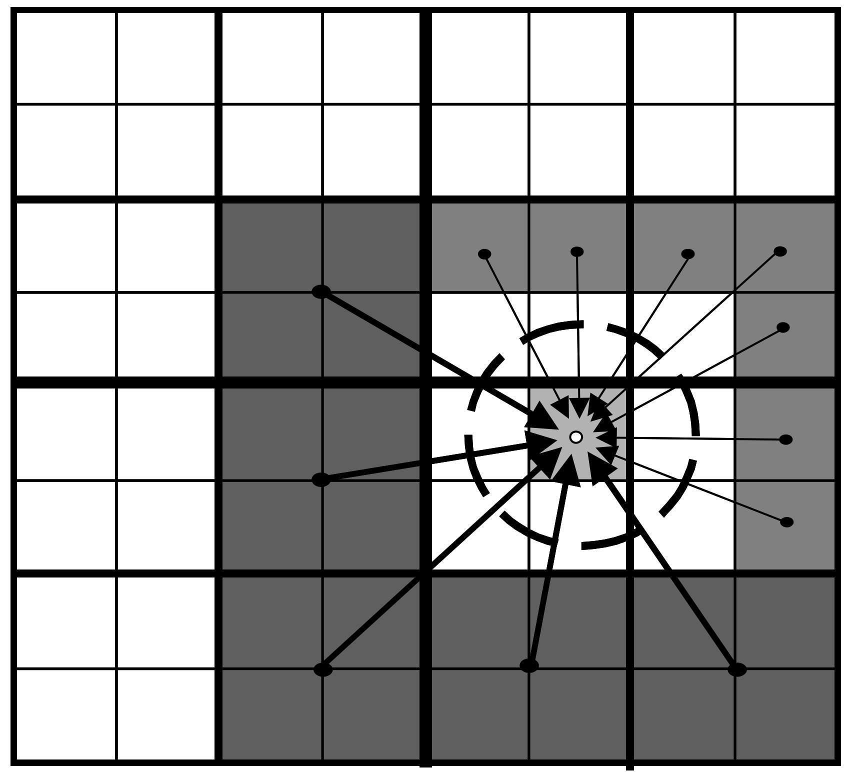
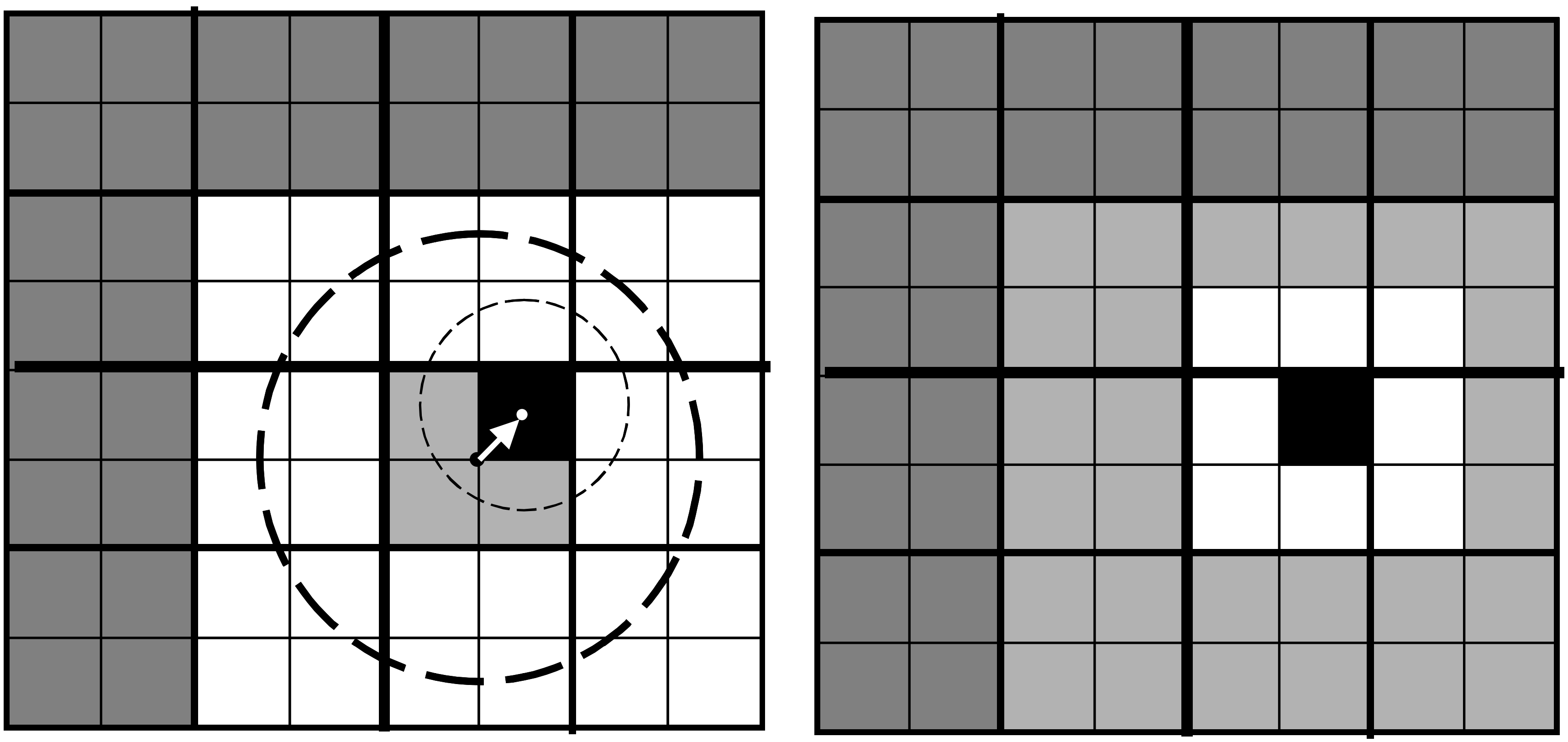


Reduced



Downward Pass. Step 2.



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