The Multilevel Fast Multipole Method

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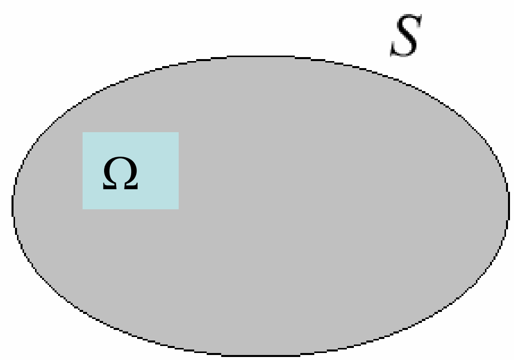
Outline

- Review
- Vector analysis (Divergence & Gradient of potential)
- 3-D Cartesian coordinates & Spherical coordinates
- Laplace’s equation and Helmholtz’ equation
- Green's function & Green's theorem
- Boundary element method
- FMM

Gauss Divergence theorem

- The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_\Omega \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot \mathbf{n} dS$$



Proof follows from the definition of divergence.

- In practice we can write

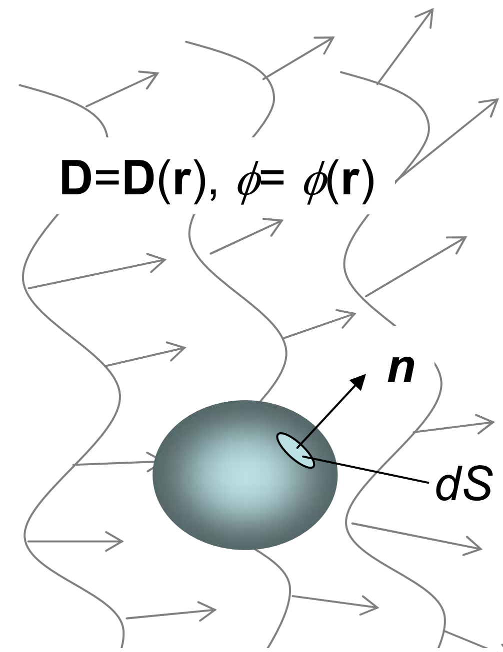
$$\int_\Omega \nabla \text{ anything} dV = \int_S \text{anything } \mathbf{n} dS$$

Integral Definitions of *div*, *grad* and *curl*

$$\nabla\phi \equiv \lim_{\delta\tau\to 0} \frac{1}{\delta\tau} \oint\!\!\!\oint_{\Delta S} \phi \mathbf{n}\, dS$$

$$\nabla \cdot \mathbf{D} \equiv \lim_{\delta\tau\to 0} \frac{1}{\delta\tau} \oint\!\!\!\oint_{\Delta S} \mathbf{D}.\mathbf{n}\, dS$$

$$\nabla \times \mathbf{D} \equiv -\lim_{\delta\tau\to 0} \frac{1}{\delta\tau} \oint\!\!\!\oint_{\Delta S} \mathbf{D} \times \mathbf{n}\, dS$$



Elemental volume $\delta\tau$ with surface ΔS

Green's formula

Green's first theorem

$$\int_\Omega (\psi \nabla^2 \phi + \nabla\phi \cdot \nabla\psi) dV = \int_S \psi \frac{\partial \phi}{\partial n} dS$$

$$\int_\Omega (\phi \nabla^2 \psi + \nabla\phi \cdot \nabla\psi) dV = \int_S \phi \frac{\partial \psi}{\partial n} dS$$

Green's second theorem (subtracting the above two)

$$\int_\Omega (\psi \nabla^2 \phi - \phi \nabla^2 \psi) dV = \int_S \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) dS$$

Laplace's equation

Let G satisfy

$$\nabla^2 G(\mathbf{x}) = \delta(\mathbf{x})$$

Solution is

$$G(\mathbf{x}) = -\frac{1}{4\pi r}$$

More generally

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|}$$

Helmholtz equation

- Let G satisfy

$$\nabla^2 G(\mathbf{x}) + k^2 G = \delta(\mathbf{x})$$

Solution is

$$G(\mathbf{x}) = -\frac{\exp(ikr)}{4\pi r}$$

More generally

$$G(\mathbf{x}, \mathbf{y}) = -\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Green's formula

$$\psi(\mathbf{x}) = \int_{S_y} \left(\psi(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y} - G(\mathbf{x},\mathbf{y}) \frac{\partial \psi}{\partial n_y}(\mathbf{y}) \right) dS_y$$

- Discretize surface S into triangles
- Discretize`

local function in terms of local isoparametric shape functions, i.e., as

$$\phi(\mathbf{x}) = \sum_{i=1}^{N} \phi_i N_i(\mathbf{x}), \quad q(\mathbf{x}) = \sum_{i=1}^{N} q_i N_i(\mathbf{x}),$$

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where usually

$$N_i(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in S_i \\ 0, & \mathbf{x} \notin S_i \end{cases} \quad \text{for constant elements.}$$

Green's formula

- Recall that the impulse-response is sufficient to characterize a linear system
- Solution to arbitrary forcing constructed via convolution
- For a linear boundary value problem we can likewise use the solution to a delta-function forcing to solve it.
- Fluid flow, steady-state heat transfer, gravitational potential, etc. can be expressed in terms of Laplace's equation $\nabla^2 P = 0 \qquad \nabla^2 P + k^2 P = 0$
- Solution to delta function forcing, without boundaries, is called free-space Green's function

$$P(\mathbf{x}) = -\int_S \left[\frac{\partial P}{\partial n}(\mathbf{y}) G(\mathbf{x},\mathbf{y}) - P(\mathbf{y}) \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) \right] dS(\mathbf{y}), \qquad \mathbf{x} \notin \Omega,$$

Boundary Element Methods

With this discretization we can write Green identity in the form

$$\frac{1}{2}\phi_j = \sum_{i=1}^{N} q_i \int_{S_i} G(\mathbf{x}-\mathbf{y}_j)\, dS(\mathbf{x}) - \sum_{i=1}^{N} \phi_i \int_{S_i} \frac{\partial G(\mathbf{x}-\mathbf{y}_j)}{\partial n_x} dS(\mathbf{x}),$$

- Boundary conditions provide value of ϕ_j or q_j
- Becomes a linear system to solve for the other

Accelerate via FMM

$$\Phi_i(\mathbf{y}) = \int_{S_i} G(\mathbf{x}-\mathbf{y})dS(\mathbf{x}),$$

$$Q_i(\mathbf{y}) = \int_{S_i} \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial n_x} dS(\mathbf{x}),$$

Using the expansion of the Green function

$$G(\mathbf{r}-\mathbf{r}_{*1}) = G(\mathbf{r}_{*1}-\mathbf{r}) = ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n} R_n^{-m}(\mathbf{r}_{*1}-\mathbf{r}_{*2})S_n^m(\mathbf{r}-\mathbf{r}_{*2}), \quad |\mathbf{r}-\mathbf{r}_{*2}| > |\mathbf{r}$$

Therefore, for such a $\mathbf{y}$ we have

$$\Phi_i(\mathbf{y}) = \int_{S_i} G(\mathbf{x}-\mathbf{y})dS(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} S_n^m(\mathbf{y}-\mathbf{x}^{(n,L)})ik\int_{S_i} R_n^{-m}(\mathbf{x}-\mathbf{x}^{(n,L)})dS(\mathbf{x}).$$

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Comparing with Eq. (ref: e1), we can determine the expansion coefficients $A_n^{(i)m}$:

$$A_n^{(i)m} = ik\int_{S_i} R_n^{-m}(\mathbf{x}-\mathbf{x}^{(n,L)})dS(\mathbf{x}).$$

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Similarly, we consider $Q_i(\mathbf{y})$. Here we note that since the triangle is flat its normal, $\mathbf{n}_i$, does not change. Therefore,

$$Q_i(\mathbf{y}) = \int_{S_i} \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial n_x} dS(\mathbf{x}) = \mathbf{n}_i \cdot \int_{S_i} \nabla_x G(\mathbf{x}-\mathbf{y})dS(\mathbf{x})$$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} S_n^m(\mathbf{y}-\mathbf{x}^{(n,L)})ik\mathbf{n}_i \cdot \int_{S_i} \nabla_x R_n^{-m}(\mathbf{x}-\mathbf{x}^{(n,L)})dS(\mathbf{x})$$

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\begin{aligned}
Q_i(\mathbf{y}) &= \int_{S_i} \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial n_x} dS(\mathbf{x}) = \mathbf{n}_i \cdot \int_{S_i} \nabla_x G(\mathbf{x}-\mathbf{y}) dS(\mathbf{x}) \\
&= -(\mathbf{n}_i \cdot \nabla_y) \int_{S_i} G(\mathbf{x}-\mathbf{y}) dS(\mathbf{x}) = -(\mathbf{n}_i \cdot \nabla_y) \Phi_i(\mathbf{y}).
\end{aligned}
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