CMSC698R/AMSC878R Mid Term Examination

1. (25 points) Below are some very short questions (mainly definitions to check if you have read the notes). Be brief. Where necessary illustrate with a rough sketch.

   (a) (2 points) Show that the number of operations needed to perform a regular product of a $N \times N$ dense matrix with a N vector requires $O(N^2)$ operations.

   (b) (3 points) Give examples of functions which have asymptotic behavior of $O(N)$, $o(N)$, and $\Theta(N)$ relative to each other.

   (c) (3 points) What is a local expansion? What is a far-field expansion? What kind of an expansion is a Taylor series?

   (d) (2 points) What is the Kronecker product of the matrices $C = A \bigotimes B$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & -5 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

   (e) (5 points) Let $\Phi(y, x)$ be a function. What are the requirements on Φ so that the sums

$$v(y_j) = \sum_{j=1}^{N} \Phi(y_j, x_i) u_i, \qquad j = 1, \ldots, M$$

   can be evaluated using the FMM?

   (f) (5 points) Explain the role of the domain E_4 in the downward pass of the multilevel fast-multipole algorithm.

   (g) (5 points) How many terms of the Taylor series do you need to approximate the function $\exp(x)$ in the range $x \in [-1, 1]$ with absolute error 10^{-8}?

2. (45 points) Consider the pre-FMM algorithm with R-expansions that was discussed in class. We use it to compute the sum

$$v(y_j) = \frac{1}{N} \sum_{i=1}^{N} \Phi(y_j, x_i) u_i, \qquad j = 1, \ldots, N$$

   The points $\{y_j\}$ and $\{x_i\}$ are distributed uniformly in a 2-D square domain of unit size. The domain is divided into K equal square boxes. It turns out that for this $\Phi(y_j, x_i)$ the number of terms, p, required for a given error, ϵ, in the R-expansion is proportional to the distance of the evaluation points from the expansion center.

   (a) Derive an expression for p as a function of K for fixed ϵ, by using the maximum distance in the evaluation box.

(b) Derive an expression for the complexity of the algorithm as a function of N and K. Assume $K \gg 1$.

(c) Determine the optimal number of boxes K as a function of N, and the overall complexity of the algorithm.

3. (30 points) Find all neighbors for box #1532 at level 5 in the oct-tree where the hierarchical numbering (indexing) with bit-interleaved coordinates is used.