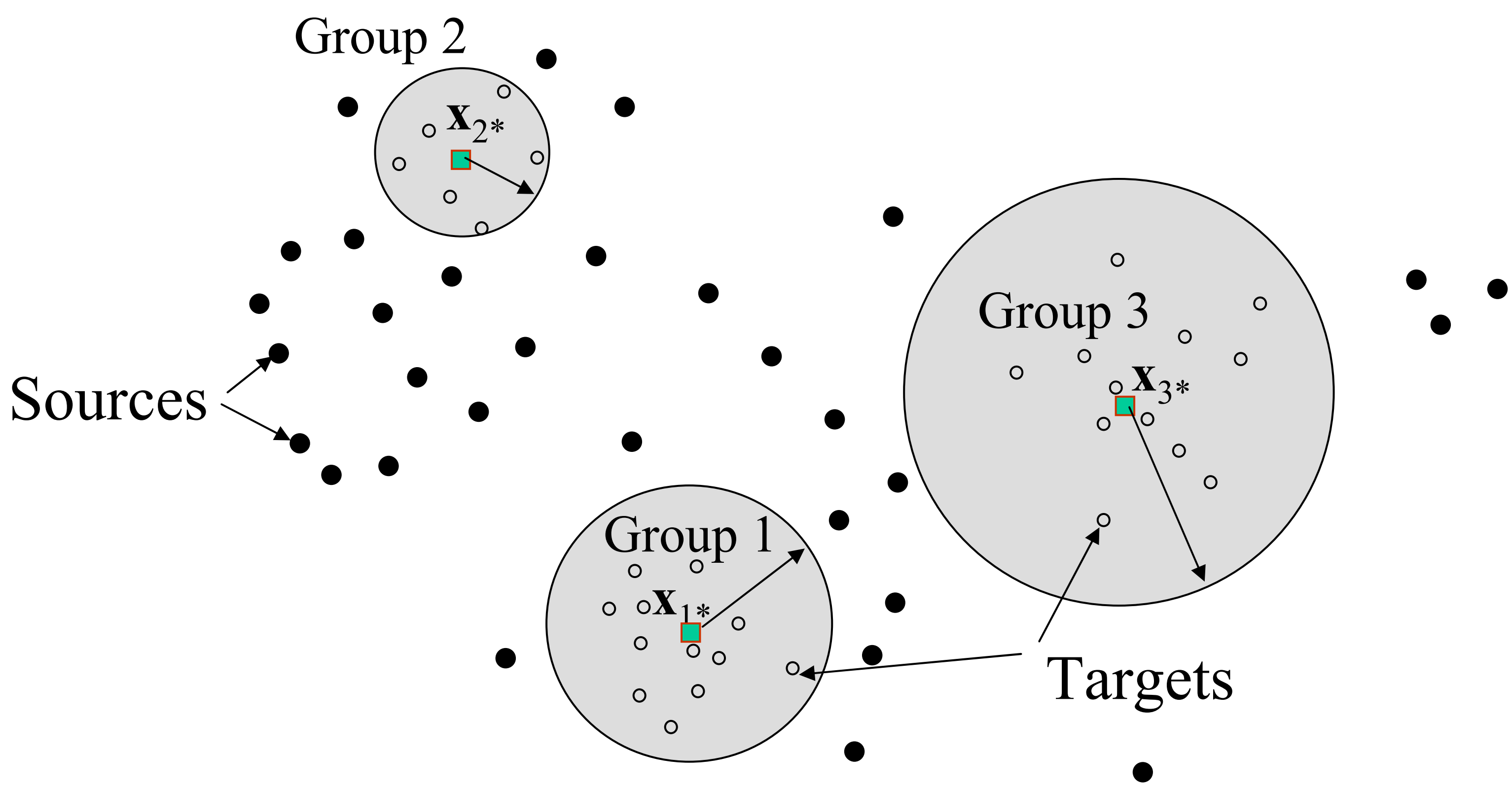
FMM CMSC 878R/AMSC 698R

Lecture 5

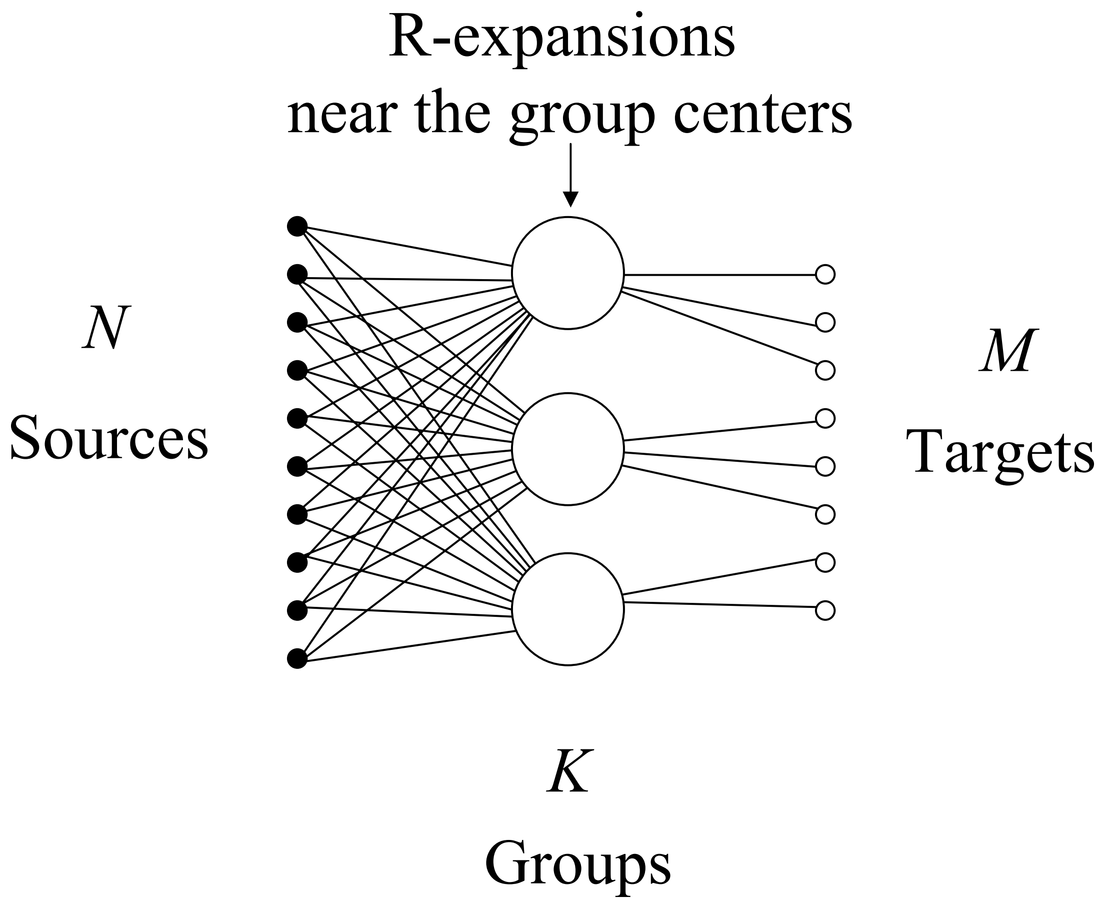
Outline

- Spatial Grouping: One of key stones of the FMM
- Natural Spatial Grouping
- Problem of “Bad Points”
- Single Level ``Pre-FMM”
- Space Partitioning with Respect to the Target Set
- Optimization of the ``Pre-FMM”
- Space Partitioning with Respect to the Source Set

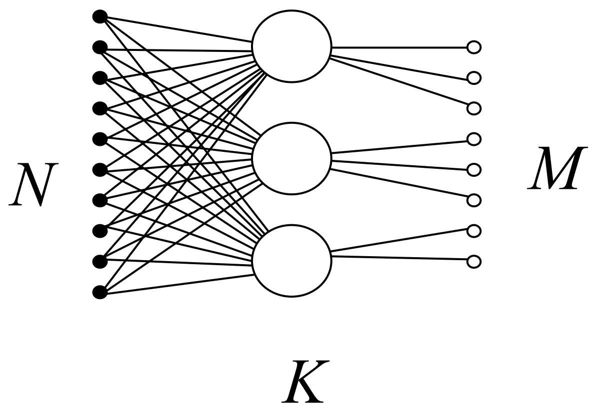
Natural Spatial Grouping for Well Separated Sets
(Grouping with respect to the Target Set)



Natural Spatial Grouping for Well Separated Sets (continuation)



Natural Spatial Grouping for Well Separated Sets (continuation)



Asymptotic Complexity:

1) Let the R-expansion has p-terms;

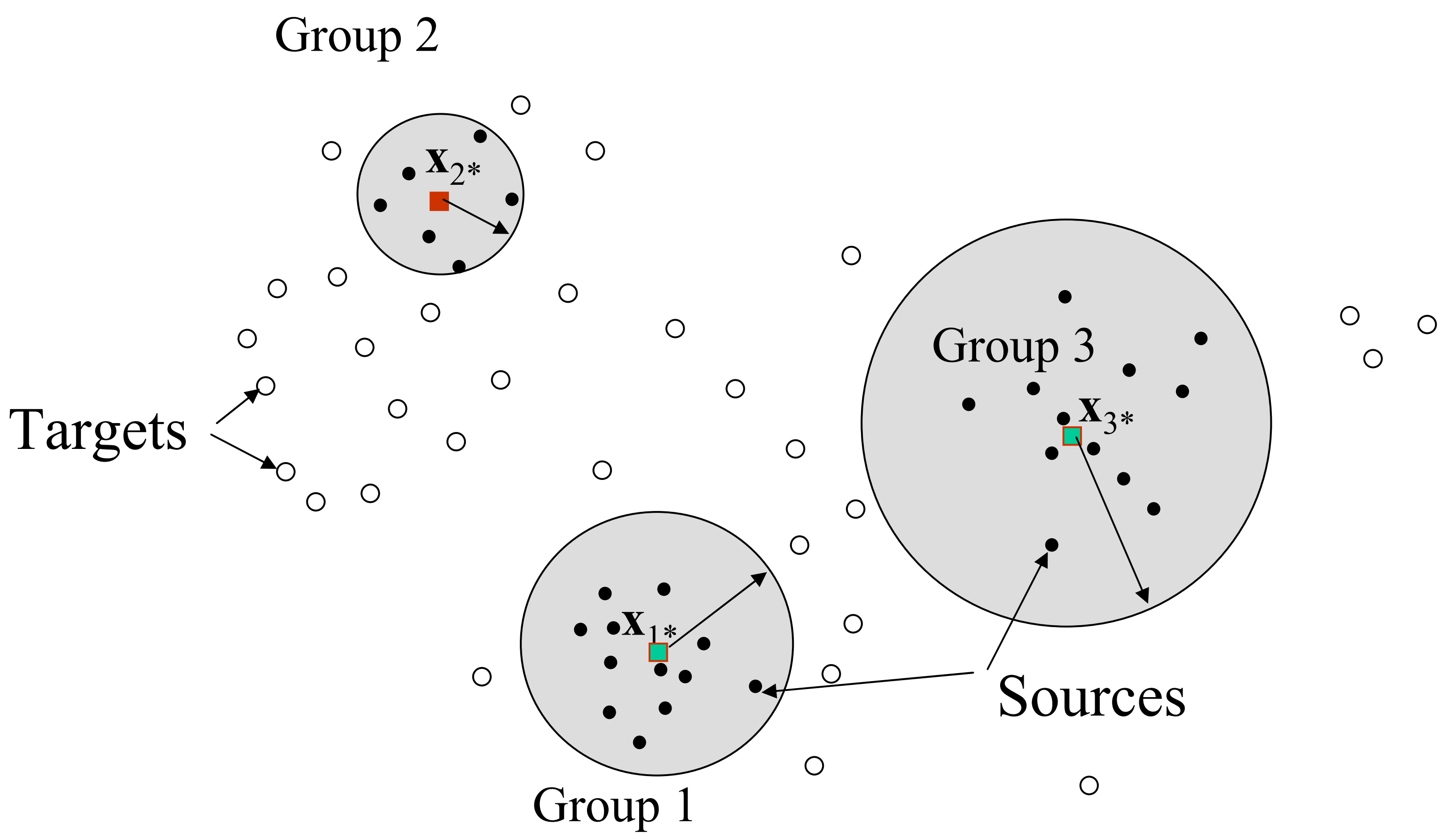
2) To build them for K groups we need $O(pNK)$ operations.

3) To evaluate them we need $O(pM)$ operations.

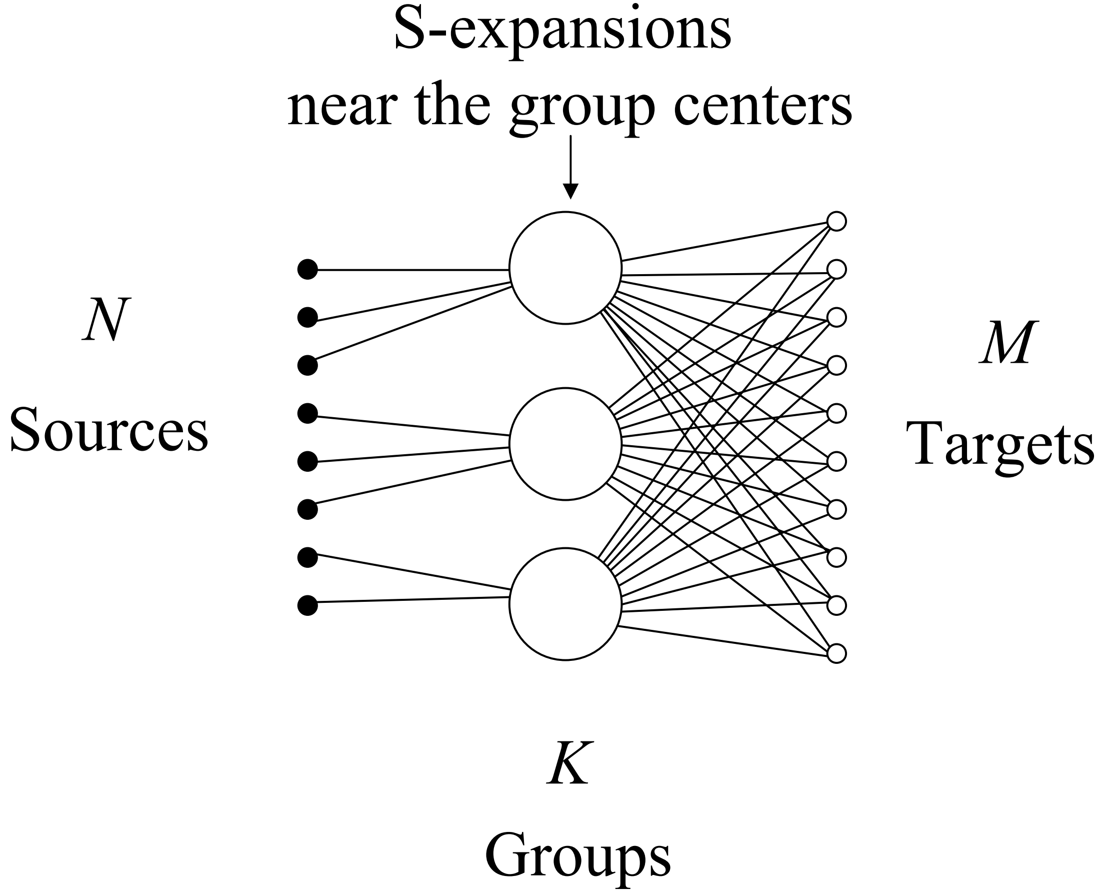
4) Total complexity: $O(p(NK+M))$.

5) Better then the Straightforward method, if $pK<<M$. In this case $p(NK+M)<<NM$

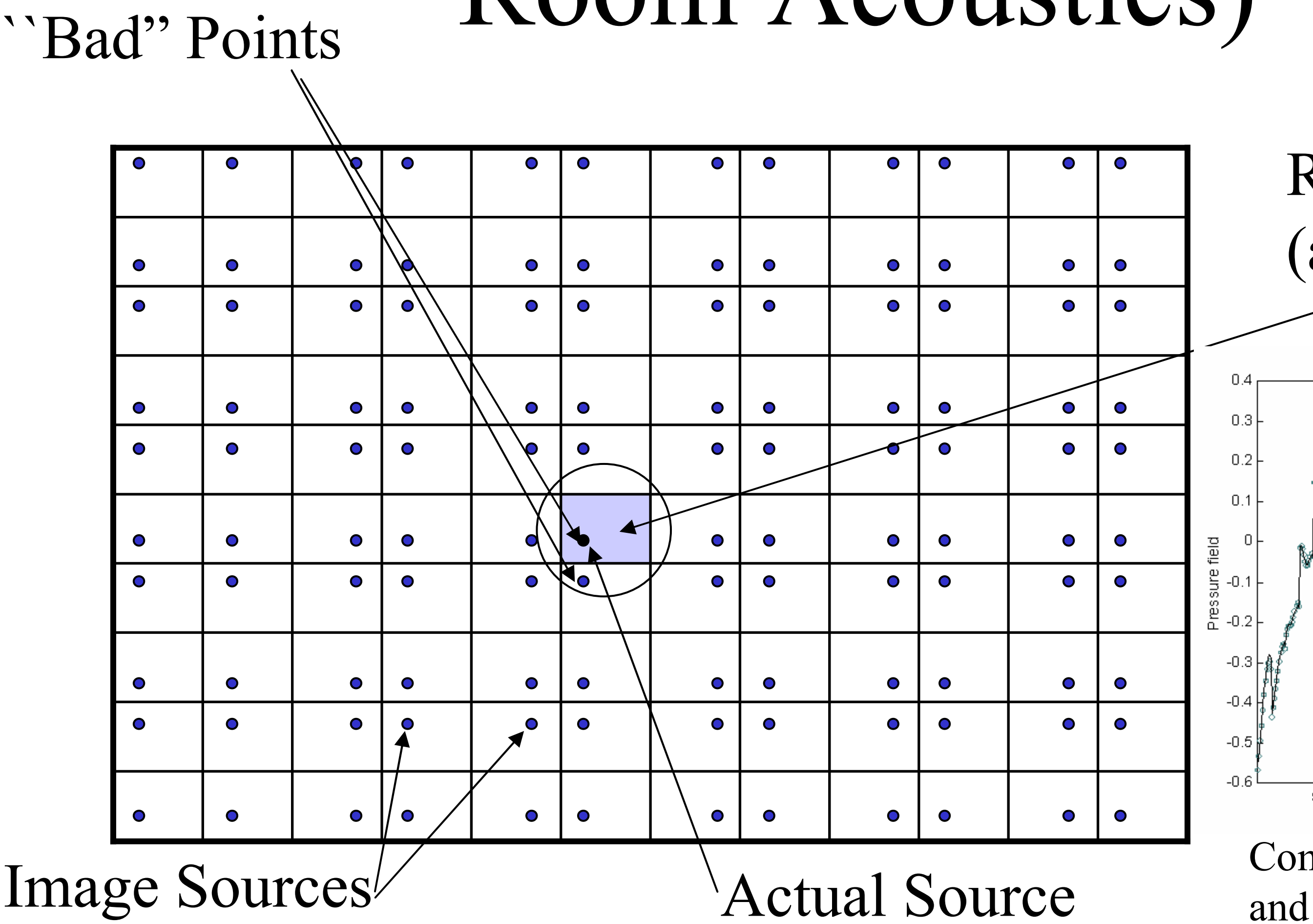
Natural Spatial Grouping for Well Separated Sets
(Grouping with respect to the Source Set)

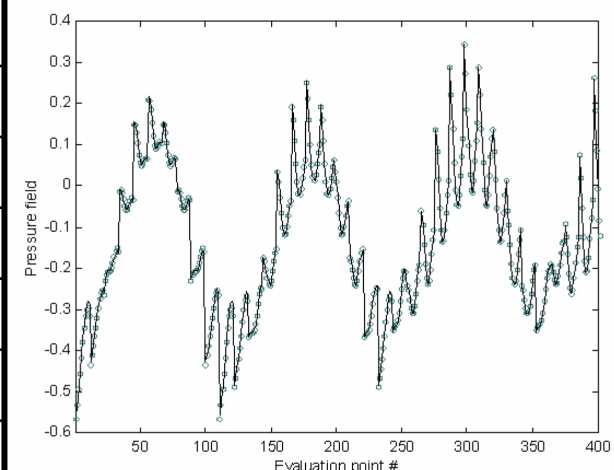


Natural Spatial Grouping for Well Separated Sets (continuation)



``Bad" Points (Example from Room Acoustics)





Comparison of Straightforward and Fast Solutions

(R. Duraiswami, N.A. Gumerov, D.N. Zotkin & L.S. Davis, Efficient Evaluation Of Reverberant Sound Fields, 2001 IEEE Workshop on Applications of Signal Processing to Audio and Acoustics, 2001).

``Bad" Points (continued)

Recipe: If the number of ``bad" points is small, compute their contribution directly.

E.g. if this number is smaller than p,
then the bad points do not change the complexity.

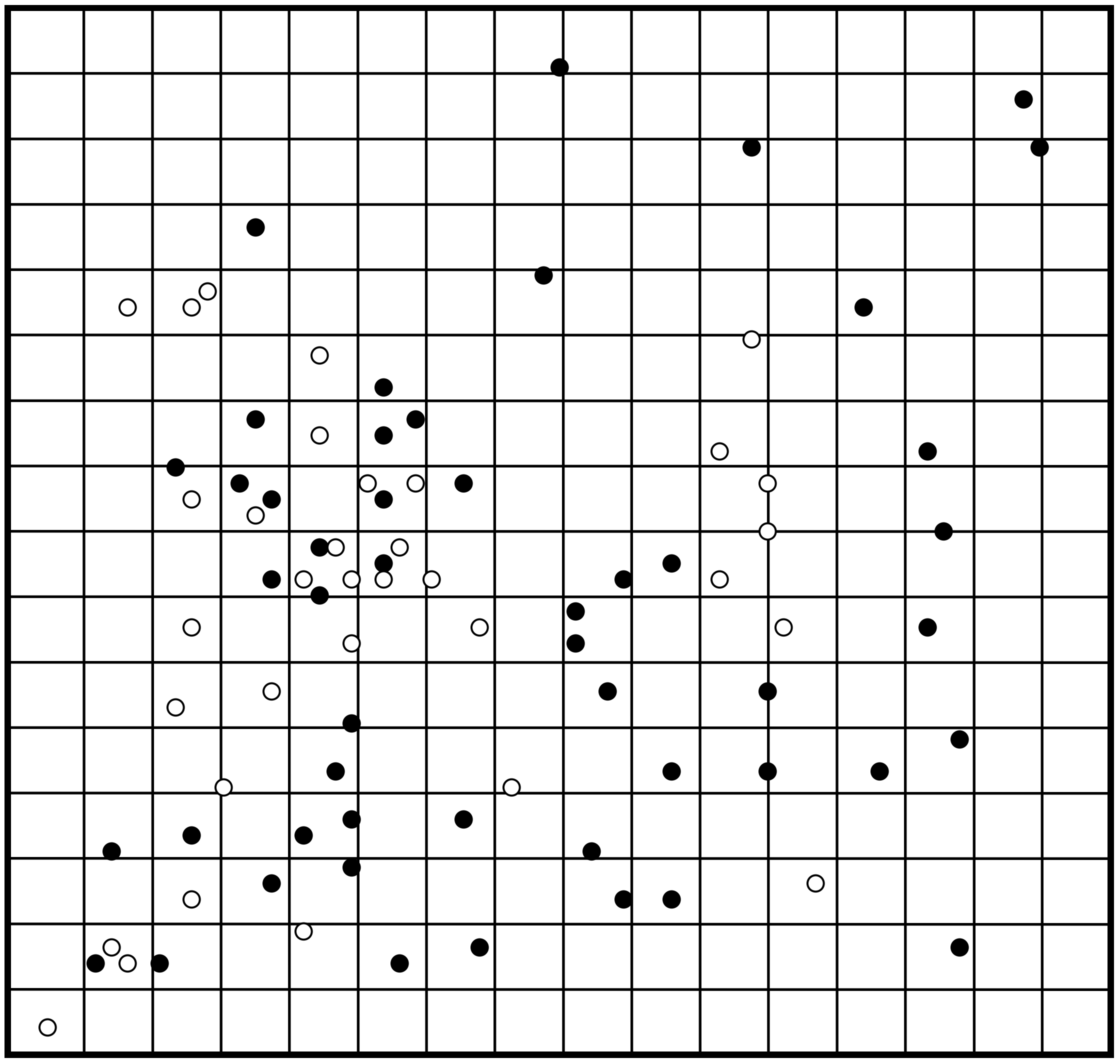
Examples of Natural Spatial Grouping

- Stars (Form Galaxies, Gravity);
- Flow Past a Body (Vortices are Grouped in a Wake);
- Statictics (Clusters of Statictical Data Points);
- People (Organized in Groups, Cities, etc.);
- Create your own example !

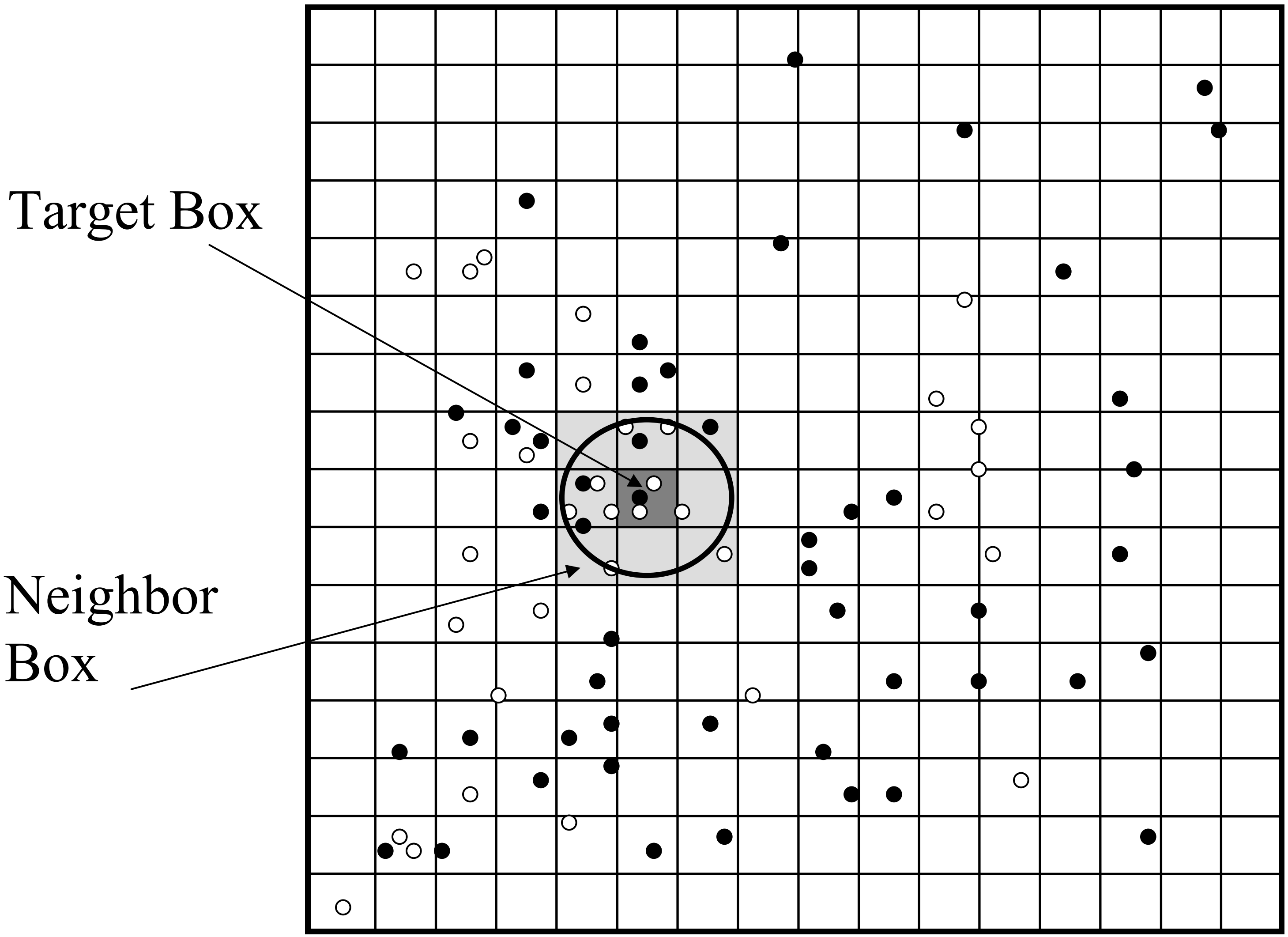
Deficiencies

- Data points may be not naturally grouped;
- Need intelligence to identify the groups: Problem with the algorithms (Artificial Intelligence?)
- Problem dependent.

The Answer Is: Space Partitioning



Space Partitioning with Respect to Target Set



An Algorithm for Computation with Space Partitioning (Pre-FMM)

- Decomposition of the sum:

Singular Part (sources in the neighborhood)

$$v(\mathbf{y}_j) = \sum_{\mathbf{x}_i \in R_n^+} u_i \Phi(\mathbf{y}_j - \mathbf{x}_i) + \sum_{\mathbf{x}_i \in R_n^-} u_i \Phi(\mathbf{y}_j - \mathbf{x}_i), \quad \mathbf{y}_j \in R_n.$$

Regular Part (sources outside the neighborhood)

- Factorization of the regular part

$$\Phi(\mathbf{y}_j - \mathbf{x}_i) = \sum_{m=0}^{p-1} a_m(\mathbf{x}_i, \mathbf{x}_{n*}) R_m(\mathbf{y}_j - \mathbf{x}_{n*}) + Error_p, \quad \mathbf{y}_j, \mathbf{x}_{n*} \in R_n, \quad \mathbf{x}_i \in R_n^-.$$

- Fast computation of the regular part

$$\sum_{\mathbf{x}_i \in R_n^-} u_i \Phi(\mathbf{y}_j - \mathbf{x}_i) = \sum_{m=0}^{p-1} \left[\sum_{\mathbf{x}_i \in R_n^-} u_i a_m(\mathbf{x}_i, \mathbf{x}_{n*}) \right] R_m(\mathbf{y}_j - \mathbf{x}_{n*}).$$

- Direct summation of the singular part, $\sum_{\mathbf{x}_i \in R_n^+} u_i \Phi(\mathbf{y}_j - \mathbf{x}_i)$

Asymptotic Complexity of the Pre-FMM

- Let N be the number of sources, M the number of targets, and K the number of target boxes.
- Each target box, R_n, M_n targets, $n = 1,\ldots,K$.
- The *neighborhood* of each target box contains N_n sources, $n = 1,\ldots,K$.
- Computation of the expansion coefficients for the regular part for the nth box requires $O((N-N_n)p)$ operations.
- Evaluation of the regular expansion for the nth box requires $O(M_n p)$ operations.
- Direct computation of the singular part requires $O(M_n N_n)$ operations.
- Total complexity is:

$$Complexity = O\left(\sum_{n=1}^{K}[(N-N_n)p + M_n p + M_n N_n]\right).$$

Asymptotic Complexity of the Pre-FMM (continued)

We have

$$\sum_{n=1}^{K} M_n = M$$

Power of the neighborhood of dimensionality *d* (the number of boxes in the neighborhood)

Consider a uniform distribution, then

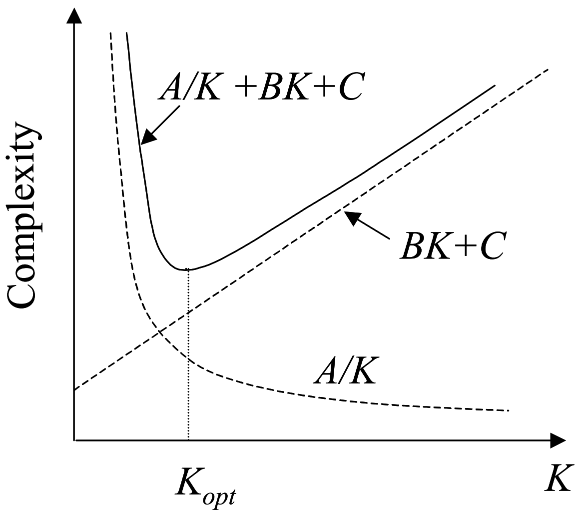
$$N_n \sim const \sim \frac{NPow(d)}{K},$$

$$F(K) = \sum_{n=1}^{K} [(N - N_n)p + M_n p + M_n N_n] = KNp - NpPow(d) + Mp + \frac{MNPow(d)}{K}$$

$$= \frac{MN}{K} Pow(d) + (K - Pow(d))Np + Mp$$

$$Complexity = O(F(K))$$

Optimization of the box number



Optimum complexity

$$F(K) = \frac{MN}{K} Pow(d) + (K - Pow(d))Np + Mp$$

$$K_{opt} = \left[\frac{MNPow(d)}{Np} \right]^{1/2} = \sqrt{\frac{MPow(d)}{p}} .$$

$$Complexity = O(F(K_{opt})) = O\left(Np\left(2\sqrt{\frac{MPow(d)}{p}} - Pow(d) \right) + Mp \right)$$

For $M \sim N,\ p \ll N$:

$$Complexity = O\left(N^{3/2} p^{1/2}\right)$$