

Lecture 4

Example

$$\Phi(\mathbf{y}, \mathbf{x}_i) = e^{\mathbf{y} \cdot \mathbf{x}_i} = \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \nabla_{\mathbf{x}_*}]^m \Phi(\mathbf{x}_*, \mathbf{x}_i),$$

Fix $(\mathbf{y} - \mathbf{x}_*)$:

$$\Phi(\mathbf{x}_*, \mathbf{x}_i) = e^{\mathbf{x}_* \cdot \mathbf{x}_i},$$

$$\nabla_{\mathbf{x}_*} \Phi(\mathbf{x}_*, \mathbf{x}_i) = \mathbf{x}_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} = \mathbf{x}_i \Phi(\mathbf{x}_*, \mathbf{x}_i),$$

$$[(\mathbf{y} - \mathbf{x}_*) \cdot \nabla_{\mathbf{x}_*}] \Phi(\mathbf{x}_*, \mathbf{x}_i) = [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i] \Phi(\mathbf{x}_*, \mathbf{x}_i),$$

$$[(\mathbf{y} - \mathbf{x}_*) \cdot \nabla_{\mathbf{x}_*}]^m \Phi(\mathbf{x}_*, \mathbf{x}_i) = [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i]^m \Phi(\mathbf{x}_*, \mathbf{x}_i),$$

$$\Phi(\mathbf{y}, \mathbf{x}_i) = \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i]^m \Phi(\mathbf{x}_*, \mathbf{x}_i) = e^{\mathbf{x}_* \cdot \mathbf{x}_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i]^m.$$

$$\text{Check: } e^{\mathbf{y} \cdot \mathbf{x}_i} = e^{\mathbf{x}_* \cdot \mathbf{x}_i} e^{(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i} = e^{\mathbf{x}_* \cdot \mathbf{x}_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i]^m.$$

Is That a Factorization?

$$e^{\mathbf{y} \cdot \mathbf{x}_i} = e^{\mathbf{x}_* \cdot \mathbf{x}_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i]^m$$

Scalar Product in d-Dimensional Space

Definition of scalar product:

$$\mathbf{a} = (a_1, \dots, a_d), \quad \mathbf{b} = (b_1, \dots, b_d),$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \dots + a_d b_d = \sum_{k=1}^d a_k b_k.$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta,$$

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

What if

$$a_1, \dots, a_d, b_1, \dots, b_d \in \mathbb{C} \quad ?$$

Definition:

$$\mathbf{a} \cdot \mathbf{b} = \overline{a_1} b_1 + \dots + \overline{a_d} b_d = \sum_{k=1}^d \overline{a_k} b_k.$$

complex
conjugate

Properties of Scalar Product

Commutativity:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

Scaling:

$$(\lambda \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b}), \quad \lambda \in \mathbb{R}$$

Distributivity:

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$$

Factorization of Scalar Product Powers

$$\begin{aligned}(\mathbf{a} \cdot \mathbf{b})^n &= \left(\sum_{k=1}^d a_k b_k \right)^n = \sum_{k_1=1}^d a_{k_1} b_{k_1} \sum_{k_2=1}^d a_{k_2} b_{k_2} \cdots \sum_{k_n=1}^d a_{k_n} b_{k_n} \\ &= \sum_{k_1=1}^d \sum_{k_2=1}^d \cdots \sum_{k_n=1}^d a_{k_1} a_{k_2} \cdots a_{k_n} b_{k_1} b_{k_2} \cdots b_{k_n} \\ &= [\mathbf{a} \otimes \mathbf{a} \otimes \cdots \otimes \mathbf{a}] \cdot [\mathbf{b} \otimes \mathbf{b} \otimes \cdots \otimes \mathbf{b}] = \mathbf{a}^n \cdot \mathbf{b}^n\end{aligned}$$

$$\mathbf{a}^n \cdot \mathbf{b}^n = (\mathbf{a} \cdot \mathbf{b})^n = (\mathbf{b} \cdot \mathbf{a})^n = \mathbf{b}^n \cdot \mathbf{a}^n.$$

$$e^{\mathbf{y} \cdot \mathbf{x}_i} = e^{\mathbf{x}_* \cdot \mathbf{x}_i} \sum_{m=0}^{\infty} \frac{1}{m!} [(\mathbf{y} - \mathbf{x}_*) \cdot \mathbf{x}_i]^m = e^{\mathbf{x}_* \cdot \mathbf{x}_i} \sum_{m=0}^{\infty} \frac{1}{m!} \mathbf{x}_i^m \cdot (\mathbf{y} - \mathbf{x}_*)^m.$$

Is That Factorization?

1) Truncation:

$$\Phi(\mathbf{y}, \mathbf{x}_i) = e^{\mathbf{y} \cdot \mathbf{x}_i} = e^{\mathbf{x}_* \cdot \mathbf{x}_i} \left[\sum_{m=0}^{p-1} \frac{1}{m!} \mathbf{x}_i^m \cdot (\mathbf{y} - \mathbf{x}_*)^m + \text{Residual}_p \right]$$

2) Fast summation:

$$\begin{aligned} v_j &= \sum_{i=1}^N u_i \Phi(\mathbf{y}_j, \mathbf{x}_i) = \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \left[\sum_{m=0}^{p-1} \frac{1}{m!} \mathbf{x}_i^m \cdot (\mathbf{y}_j - \mathbf{x}_*)^m + \text{Residual}_p \right] \\ &= \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \sum_{m=0}^{p-1} \frac{1}{m!} \mathbf{x}_i^m \cdot (\mathbf{y}_j - \mathbf{x}_*)^m + N \max_i (u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i}) \text{Residual}_p \\ &= \sum_{m=0}^{p-1} \frac{1}{m!} \left(\sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \mathbf{x}_i^m \right) \cdot (\mathbf{y}_j - \mathbf{x}_*)^m + \text{Residual} \\ &= \sum_{m=0}^{p-1} \mathbf{c}_m \cdot (\mathbf{y}_j - \mathbf{x}_*)^m + \text{Residual}, \quad \mathbf{c}_m = \frac{1}{m!} \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \mathbf{x}_i^m. \end{aligned}$$

Yes! It is!

Complexity of Product

$$\mathbf{a} = (a_1, a_2),$$

$$\mathbf{a}^2 = (a_1(a_1, a_2), a_2(a_1, a_2)) = (a_1^2, a_1a_2, a_2a_1, a_2^2),$$

$$\begin{aligned} \mathbf{a}^3 &= (a_1^2(a_1, a_2), a_1a_2(a_1, a_2), a_2a_1(a_1, a_2), a_2^2(a_1, a_2)) \\ &= (a_1^3, a_1^2a_2, a_1a_2a_1, a_1a_2^2, a_2a_1^2, a_2a_1a_2, a_2^2a_1, a_2^3), \dots \end{aligned}$$

The length of \mathbf{a}^n is $2^n!$  This is not factorial!

In d dimensions the length of \mathbf{a}^n is even d^n

What to do in practical problems?

Use Compression!

Compression operator:

$$\mathbf{A}^n = \text{Compress}(\mathbf{a}^n)$$

Required Property:

$$\mathbf{a}^n \cdot \mathbf{b}^n = \text{Compress}(\mathbf{a}^n) \cdot \text{Compress}(\mathbf{b}^n).$$

Consider \mathbf{R}^2 :

$$\begin{aligned}\mathbf{a}^n \cdot \mathbf{b}^n &= (\mathbf{a} \cdot \mathbf{b})^n = (a_1 b_1 + a_2 b_2)^n \\ &= a_1^n b_1^n + \binom{n}{1} a_1^{n-1} b_1^{n-1} a_2 b_2 + \binom{n}{2} a_1^{n-2} b_1^{n-2} a_2^2 b_2^2 + \dots + a_2^n b_2^n\end{aligned}$$

The length is only $(n+1)$, not 2^n

Let us define:

$$\mathbf{A}^n = \text{Compress}(\mathbf{a}^n) = \left(a_1^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_2, \sqrt{\binom{n}{2}} a_1^{n-2} a_2^2, \dots, a_2^n \right),$$

$$\mathbf{B}^n = \text{Compress}(\mathbf{b}^n) = \left(b_1^n, \sqrt{\binom{n}{1}} b_1^{n-1} b_2, \sqrt{\binom{n}{2}} b_1^{n-2} b_2^2, \dots, b_2^n \right)$$

Example of Fast Computation

$$v_j = \sum_{i=1}^N u_i \Phi(\mathbf{y}_j, \mathbf{x}_i) = \sum_{m=0}^{p-1} \mathbf{c}_m \cdot (\mathbf{y}_j - \mathbf{x}_*)^m + \text{Residual}, \quad \mathbf{c}_m = \frac{1}{m!} \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \mathbf{x}_i^m.$$

Equivalent to:

$$v_j = \sum_{m=0}^{p-1} \mathbf{C}_m \cdot \mathbf{Compress}((\mathbf{y}_j - \mathbf{x}_*)^m) + \text{Residual}, \quad \mathbf{C}_m = \frac{1}{m!} \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \mathbf{Compress}(\mathbf{x}_i^m).$$

Number of multiplications (complexity) to obtain v_j :

$$\text{Complexity} = 1 + 2 + \dots + p = \frac{p(p+1)}{2}.$$

Compression Can be Performed for any Dimensionality (Example for 3D):

$$\mathbf{a}^n \cdot \mathbf{b}^n = (\mathbf{a} \cdot \mathbf{b})^n = (a_1 b_1 + a_2 b_2 + a_3 b_3)^n$$

$$= [(a_1 b_1 + a_2 b_2) + a_3 b_3]^n = \sum_{m=0}^n \binom{n}{m} (a_1 b_1 + a_2 b_2)^{n-m} a_3^m b_3^m$$

$$= \sum_{m=0}^n \sum_{l=0}^{n-m} \binom{n}{m} \binom{n-m}{l} a_1^{n-m-l} b_1^{n-m-l} a_2^l b_2^l a_3^m b_3^m$$

$$= a_1^n b_1^n + \binom{n}{1} a_1^{n-1} b_1^{n-1} a_2 b_2 + \binom{n}{2} a_1^{n-2} b_1^{n-2} a_2^2 b_2^2 + \dots + a_2^n b_2^n$$

$$+ \binom{n}{1} a_1^{n-1} b_1^{n-1} a_3 b_3 + \binom{n}{1} \binom{n-1}{1} a_1^{n-2} b_1^{n-2} a_2 b_2 a_3 b_3 + \dots + a_3^n b_3^n$$

$$\text{Compress}(\mathbf{a}^n) = \left(a_1^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_2, \sqrt{\binom{n}{2}} a_1^{n-2} a_2^2, \dots, a_2^n, \sqrt{\binom{n}{1}} a_1^{n-1} a_3, \dots, a_3^n \right)$$

The length of \mathbf{a}^n is $(n+1)+n+\dots+1 = (n+1)(n+2)/2$

Compression Can be Performed for any Dimensionality (General Case):

$$(a_1 + a_2 + \dots + a_d)^n = \sum_{n_1 + \dots + n_d = n} (n, n_1, n_2, \dots, n_d) a_1^{n_1} a_2^{n_2} \dots a_d^{n_d}.$$

$$(n, n_1, n_2, \dots, n_d) = \frac{n!}{n_1! n_2! \dots n_d!}.$$

Multinomial
coefficients

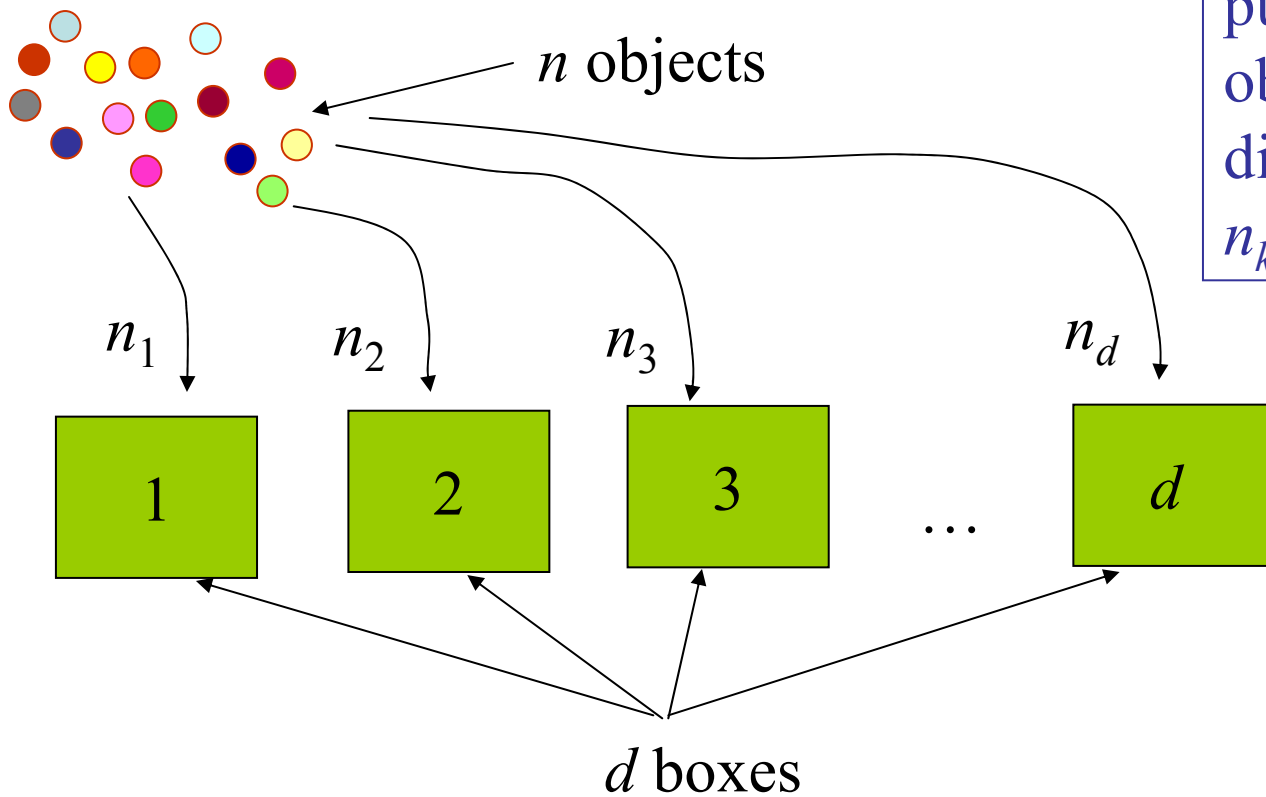
$$\text{Compress}(\mathbf{a}^n) = \left(a_1^n, \sqrt{(n, n-1, 1, 0, \dots, 0)} a_1^{n-1} a_2, \dots, \sqrt{(n, n_1, n_2, \dots, n_d)} a_1^{n_1} a_2^{n_2} \dots a_d^{n_d}, \dots, a_d^n \right)$$

So we have

$$\begin{aligned} \mathbf{a}^n \cdot \mathbf{b}^n &= \text{Compress}(\mathbf{a}^n) \cdot \text{Compress}(\mathbf{b}^n) \\ &= \sum_{n_1 + \dots + n_d = n} (n, n_1, n_2, \dots, n_d) a_1^{n_1} a_2^{n_2} \dots a_d^{n_d} b_1^{n_1} b_2^{n_2} \dots b_d^{n_d} \\ &= (a_1 b_1 + a_2 b_2 + \dots + a_d b_d)^n = (\mathbf{a} \cdot \mathbf{b})^n. \end{aligned}$$

What are multinomial coefficients?

$(n ; n_1, n_2, \dots, n_d)$ is the number of ways of putting n different objects into d different boxes with n_k in the k -th box



$$n_1 + n_2 + \dots + n_d = n$$

The length of the compressed vector

$$d = 1 : 1,$$

$$d = 2 : n + 1,$$

$$d = 3 : \frac{1}{2}(n + 1)(n + 2),$$

...

Theorem: If $\mathbf{a} \in \mathbb{R}^d$, then the length of compressed vector $\mathbf{Compress}(\mathbf{a}^n)$, is

$$\binom{n + d - 1}{n} = \frac{(n + 1) \dots (n + d - 1)}{(d - 1)!}.$$

Proof: We have a basis for induction (see above). Let this holds for d dimensions.
Consider $d + 1$ dimensions:

$$((a_1 + \dots + a_d) + a_{d+1})^n = \sum_{m=0}^n \binom{n}{m} (a_1 + \dots + a_d)^m a_{d+1}^{n-m}$$

The number of terms is then

$$\sum_{m=0}^n \binom{m + d - 1}{m} = \binom{d - 1}{0} + \binom{d}{1} + \dots + \binom{n + d - 1}{n} = \binom{n + d}{n}$$

This proves the theorem.

Example of Fast Computation

$$v_j = \sum_{i=1}^N u_i \Phi(\mathbf{y}_j, \mathbf{x}_i) = \sum_{m=0}^{p-1} \mathbf{c}_m \cdot (\mathbf{y}_j - \mathbf{x}_*)^m + \text{Residual}, \quad \mathbf{c}_m = \frac{1}{m!} \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \mathbf{x}_i^m.$$

Equivalent to:

$$v_j = \sum_{m=0}^{p-1} \mathbf{C}_m \cdot \text{Compress}((\mathbf{y}_j - \mathbf{x}_*)^m) + \text{Residual}, \quad \mathbf{C}_m = \frac{1}{m!} \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} \text{Compress}(\mathbf{x}_i^m).$$

Number of multiplications (complexity) to obtain v_j : (in 2D case!)

$$\text{Complexity} = 1 + 2 + \dots + p = \frac{p(p+1)}{2}.$$

$$\mathbf{C}_0 = \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i},$$

$$\mathbf{C}_1 = (C_{11}, C_{12}) = \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} (x_{i1}, x_{i2}),$$

$$\mathbf{C}_2 = (C_{21}, C_{22}, C_{23}) = \sum_{i=1}^N u_i e^{\mathbf{x}_* \cdot \mathbf{x}_i} (x_{i1}^2, \sqrt{2} x_{i1} x_{i2}, x_{i2}^2),$$

- $C(p+d-1, d)$ is asymptotically d^p for large d and fixed p

$$(p+d-1)! / (p-1!)d!$$

Recall Stirling's formula $n! \sim \sqrt{2\pi n} (n + 1/2) e^{-n}$

Complexity of Fast Summation

Let \circ be a scalar product of vectors A_i and F_j of length $P(p)$ (p is the truncation number).

Complexity of summation over i is then $O(PN)$.

Complexity of scalar product operation is P .

Complexity of M scalar product operations is $O(PM)$ (for $j = 1, \dots, M$).

Total complexity is $O(PM + PN)$.

Fast Method is more efficient than direct only if $O(PM + PN) < O(MN)$,
so we should have

$$P(p) \ll \min(M, N)$$

General Forms of Factorization for Fast Summation (1)

$$v_j = \sum_{i=1}^N u_i \Phi(y_j, x_i), \quad j = 1, \dots, M.$$

$$\begin{aligned} \Phi(y_j, x_i) &= \sum_{m=0}^p a_m(x_i, x_*) f_m(y_j - x_*) + \text{Error}(p, x_i, x_*, y_j) \\ &= \mathbf{a}(x_i, x_*) \cdot \mathbf{f}(y_j - x_*) + \text{Error}. \end{aligned}$$

scalar product

How about

vectors of length p

$$v_j = \sum_{i=1}^N u_i e^{-\lambda_i |x_i - y_j|^2}$$

Some parameter depending on i

More general to have

$$v_j = \sum_{i=1}^N u_i \Phi_i(y_j) \quad \text{or} \quad v(\mathbf{y}) = \sum_{i=1}^N u_i \Phi_i(\mathbf{y}).$$

General Forms of Factorization for Fast Summation (2)

The potential can be factorized as

$$\Phi_i(\mathbf{y}) = \mathbf{A}_i(\mathbf{x}_*) \circ \mathbf{F}(\mathbf{y} - \mathbf{x}_*)$$

Generalized product \circ can be scalar product, contraction, etc. \mathbf{A}_i and \mathbf{F} can be real or complex vectors, tensors, etc. in p -dimensional space.

Requirements to the product (distributivity with respect to addition)

$$(\alpha\mathbf{A}_i + \beta\mathbf{A}_j) \circ \mathbf{F} = \alpha\mathbf{A}_i \circ \mathbf{F} + \beta\mathbf{A}_j \circ \mathbf{F}.$$

In this case

$$v(\mathbf{y}) = \sum_{i=1}^N u_i \Phi_i(\mathbf{y}) = \sum_{i=1}^N u_i \mathbf{A}_i(\mathbf{x}_*) \circ \mathbf{F}(\mathbf{y} - \mathbf{x}_*) = \mathbf{A}(\mathbf{x}_*) \circ \mathbf{F}(\mathbf{y} - \mathbf{x}_*)$$

$$\mathbf{A}(\mathbf{x}_*) = \sum_{i=1}^N u_i \mathbf{A}_i(\mathbf{x}_*)$$

We do not need commutativity of \circ (i.e. we do not request $\mathbf{A}_i \circ \mathbf{F} = \mathbf{F} \circ \mathbf{A}_i$)(!).

General Forms of Factorization for Fast Summation (3)

Actually, we even do need continuous variable \mathbf{y} ,
The problem is to represent all matrix elements in the form

$$\Phi_{ji} = \mathbf{A}_i \circ \mathbf{F}_j$$

then

$$\mathbf{v}_j = \sum_{i=1}^N u_i \Phi_{ji} = \sum_{i=1}^N u_i (\mathbf{A}_i \circ \mathbf{F}_j) = \left(\sum_{i=1}^N u_i \mathbf{A}_i \right) \circ \mathbf{F}_j.$$

Complexity of Fast Summation

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Total complexity is $O(PM + PN)$.

Fast Method is more efficient than direct only if $O(PM + PN) < O(MN)$,

so we should have

$$P(p) \ll \min(M, N)$$

Outline

- Far Field Expansions (or S-expansions)
 - Regular Potential (Convergent Series);
 - Regular Potential (Asymptotic Series);
 - Singular Potential;
- Asymptotic Series
- Approaches for Selection of the Basis Functions

Far Field Expansions (S-expansions)

Let

$$\mathbf{x}_* \in \mathbb{R}^d.$$

Might be
Singular (at $\mathbf{y} = \mathbf{x}_*$)
Basis Functions

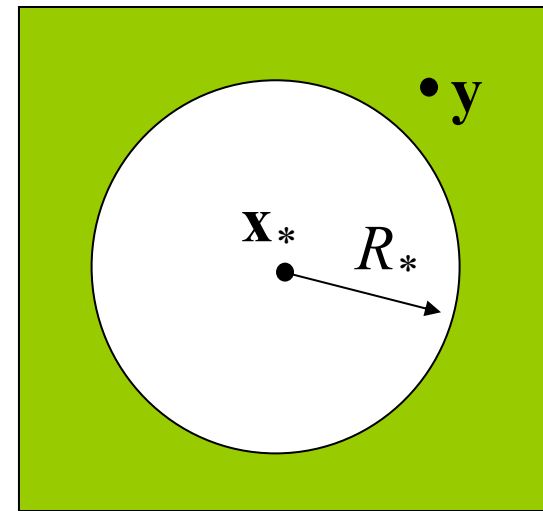
We call expansion

$$\Phi(\mathbf{y}, \mathbf{x}_i) = \sum_{m=0}^{\infty} b_m(\mathbf{x}_i, \mathbf{x}_*) S_m(\mathbf{y} - \mathbf{x}_*)$$

far field expansion (or S-expansion) outside a sphere

$$|\mathbf{y} - \mathbf{x}_*| > R_*,$$

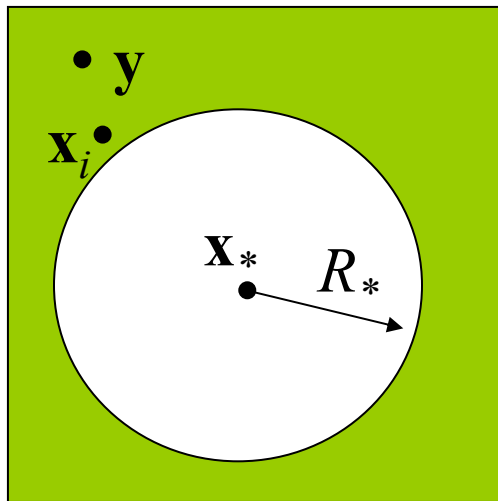
if the series converges for $\forall \mathbf{y}, |\mathbf{y} - \mathbf{x}_*| > R_*$.



Far Field Expansion of a Regular Potential

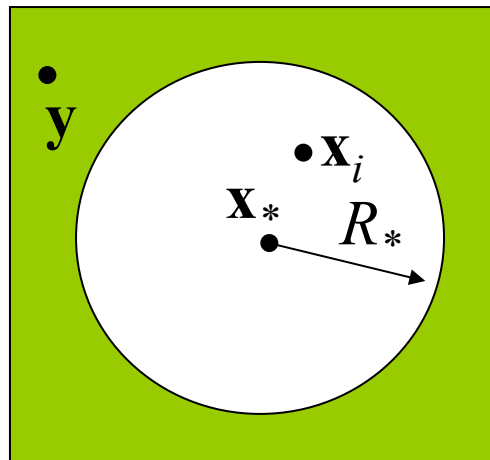
...sometimes like this:

...sometimes like this:

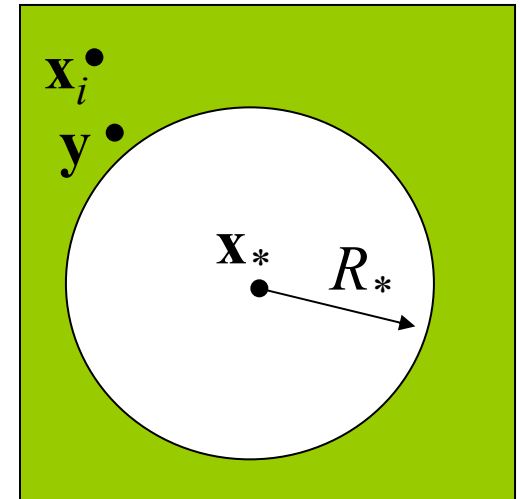


$$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$$

Can be like this:



$$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$$



$$|\mathbf{x}_i - \mathbf{x}_*| > |\mathbf{y} - \mathbf{x}_*| > R_*$$

Local Expansion of a Regular Potential Can be Far Field Expansion Also (Repeat Example from Lecture 3)

Valid for any $r_* < \infty$, and x_i

$$\Phi(y, x_i) = e^{-(y-x_i)^2} = \sum_{m=0}^{\infty} a_m(x_i, x_*) S_m(y-x_*).$$

We have

$$\begin{aligned} e^{-(y-x_i)^2} &= e^{-[y-x_*(x_i-x_*)]^2} = e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} e^{2(x_i-x_*)(y-x_*)} \\ &= e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} \sum_{m=0}^{\infty} \frac{2^m (x_i-x_*)^m (y-x_*)^m}{m!}. \end{aligned}$$

Choose

$$\begin{aligned} a_m(x_i, x_*) &= e^{-(x_i-x_*)^2} (x_i-x_*)^m, \quad m = 0, 1, \dots, \\ S_m(y-x_*) &= e^{-(y-x_*)^2} \frac{2^m}{m!} (y-x_*)^m, \quad m = 0, 1, \dots \end{aligned}$$

Asymptotic Series

$$f(x; \epsilon) = f_0(x)\varphi_0(\epsilon) + f_1(x)\varphi_1(\epsilon) + f_2(x)\varphi_2(\epsilon) + \dots = \sum_{n=0}^{\infty} f_n(x)\varphi_n(\epsilon)$$

$$\lim_{\epsilon \rightarrow 0} \frac{\varphi_n(\epsilon)}{\varphi_{n+1}(\epsilon)} = 0.$$

Gauge
functions



$$f(x; \epsilon) - \sum_{n=0}^{p-1} f_n(x)\varphi_n(\epsilon) = O(f_p(x)\varphi_p(\epsilon))$$

The asymptotic expansion is *uniform* in domain $x \in \Omega$ if

$$\forall x \in \Omega, \quad \left| f(x; \epsilon) - \sum_{n=0}^{p-1} f_n(x)\varphi_n(\epsilon) \right| = O(\varphi_p(\epsilon)).$$

Otherwise the asymptotic expansion is not uniform.

Examples of Uniform and Non-Uniform Expansions

Example of uniform expansion:

$$f(x, \epsilon) = \frac{1}{x + \epsilon}, \quad x > 10$$

$$f(x, \epsilon) = \frac{1}{x} \left(1 + \frac{\epsilon}{x}\right)^{-1} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n \epsilon^n}{x^n}$$

Example of non-uniform expansion:

$$f(x, \epsilon) = e^{\epsilon x}, \quad x \in \mathbb{R}^1$$

$$e^{\epsilon x} = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}.$$

Prove that! (Hint: consider $x \gg \epsilon^{-1}$).

Example of Far Field Expansion of a Regular Function (Using Asymptotic Series)

$$\Phi(y, x_i) = \frac{1}{1 + (y - x_i)^2} = \frac{1}{1 + [y - x_* - (x_i - x_*)]^2} = \frac{1}{(y - x_*)^2} \frac{(y - x_*)^2}{1 + [y - x_* - (x_i - x_*)]^2}.$$

Let

$$\epsilon = \frac{1}{y - x_*}$$

$$\Phi(\epsilon, x_i - x_*) = \epsilon^2 \frac{\frac{1}{\epsilon^2}}{1 + \left[\frac{1}{\epsilon} - (x_i - x_*)\right]^2} = \epsilon^2 \frac{1}{\epsilon^2 + (1 - \epsilon x)^2} = \epsilon^2 f(x, \epsilon), \quad x = x_i - x_*$$

$$f(x, \epsilon) = \frac{1}{\epsilon^2 + (1 - \epsilon x)^2} = \sum_{n=0}^{\infty} f_n(x) \epsilon^n$$

$$f_n(x) = \frac{1}{n!} \left. \frac{\partial^n f(x, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0}$$

Example of Far Field Expansion of a Regular Function (continuation)

$$f_0(x) = 1,$$

$$f_1(x) = \left. \frac{\partial f(x, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} = 2x,$$

$$f_2(x) = \left. \frac{1}{2!} \frac{\partial^2 f(x, \epsilon)}{\partial \epsilon^2} \right|_{\epsilon=0} = 3x^2 - 1,$$

$$\Phi(y, x_i) = \frac{1}{(y - x_*)^2} \sum_{n=0}^{\infty} f_n(x_i - x_*) \frac{1}{(y - x_*)^n}$$

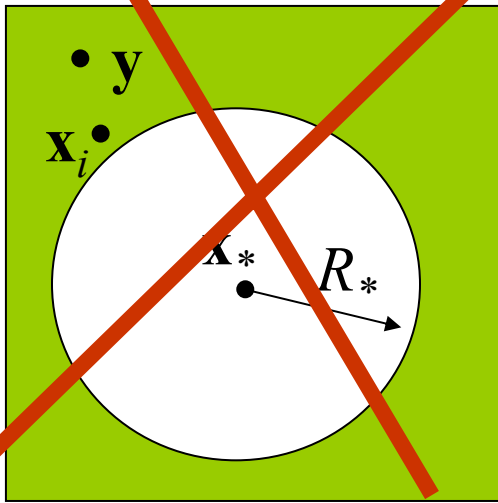
$$y \geq 100, \quad x_i = 1, \quad x_* = 0,$$

$$\epsilon = 10^{-2}, \quad x = 1$$

$$\left| \Phi(y, x_i) - \frac{1}{(y - x_*)^2} \left[1 + \frac{2(x_i - x_*)}{(y - x_*)} \right] \right| \leq \epsilon^4 (3x^2 - 1) = 2 \cdot 10^{-8}.$$

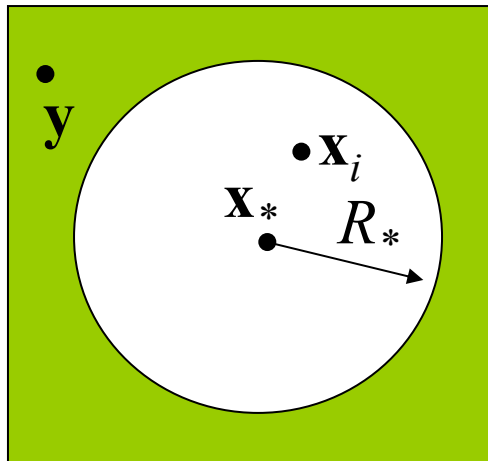
Far Field Expansion of a Singular Potential

...sometimes like this:



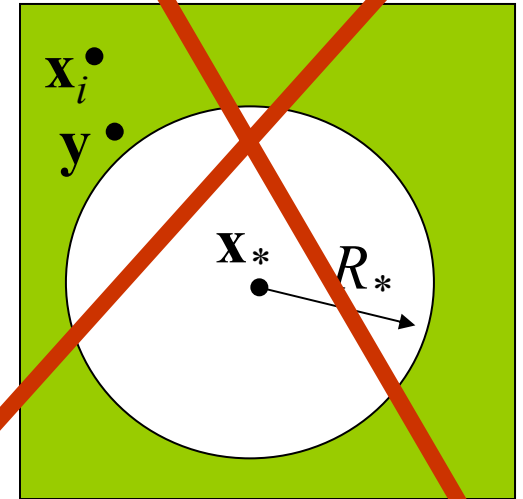
$$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$$

Can be like this:



$$|\mathbf{y} - \mathbf{x}_*| > R_* \geq |\mathbf{x}_i - \mathbf{x}_*|$$

...sometimes like this:



$$|\mathbf{x}_i - \mathbf{x}_*| > |\mathbf{y} - \mathbf{x}_*| > R_*$$

This case only!

Example For S-expansion of Singular Potential

$$\Phi(y, x_i) = \frac{1}{y - x_i}.$$

$$\frac{1}{y - x_i} = \frac{1}{y - x_* - (x_i - x_*)} = \frac{1}{(y - x_*) \left[1 - \frac{x_i - x_*}{y - x_*} \right]} = \frac{1}{(y - x_*)} \left[1 - \frac{x_i - x_*}{y - x_*} \right]^{-1}.$$

$$\left[1 - \frac{x_i - x_*}{y - x_*} \right]^{-1} = \sum_{m=0}^{\infty} \frac{(x_i - x_*)^m}{(y - x_*)^m}, \quad |y - x_*| > |x_i - x_*|.$$

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*),$$

$$b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \dots,$$

$$S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \dots$$

Let us compare with the R-expansion of the same function

$$|y - x_*| < |x_i - x_*| :$$

R-expansion

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_*) R_m(y - x_*),$$

$$a_m(x_i, x_*) = -(x_i - x_*)^{-m-1}, \quad m = 0, 1, \dots,$$

$$R_m(y - x_*) = (y - x_*)^m, \quad m = 0, 1, \dots$$

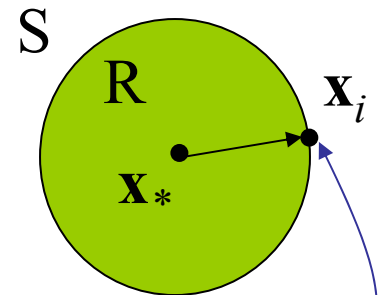
$$|y - x_*| > |x_i - x_*| :$$

S-expansion

$$\Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_*) S_m(y - x_*),$$

$$b_m(x_i, x_*) = (x_i - x_*)^m, \quad m = 0, 1, \dots,$$

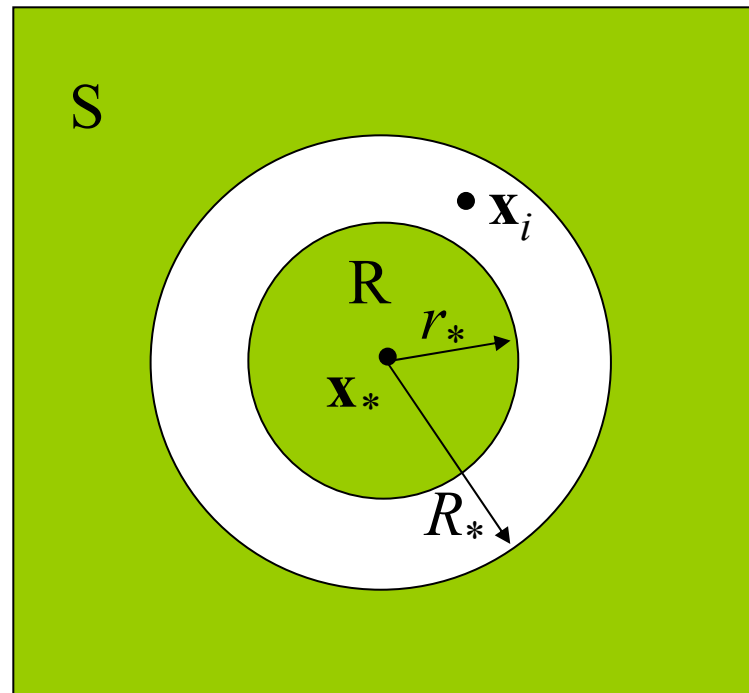
$$S_m(y - x_*) = (y - x_*)^{-m-1}, \quad m = 0, 1, \dots$$



Singular Point is located at the Boundary of regions for the R- and S-expansions!

What Do We Need For Real FMM

(that provides spatial grouping)



$$r_* < R_*$$

We need S-expansion for $|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$
We need R-expansion for $|\mathbf{y} - \mathbf{x}_*| < r_* < |\mathbf{x}_i - \mathbf{x}_*|$

Basis Functions

- Power series are great, but do they provide the best approximation? (sometimes yes!)
- Other approaches to factorization:
 - Asymptotic Series (Can be divergent!);
 - Orthogonal Bases in L_2 ;
 - Eigen Functions of Differential Operators;
 - Functions Generated by Differentiation or Other Linear Operators.
- Some of these approaches will be considered in this course.