FMM CMSC 878R/AMSC 698R

Lecture 17

Outline

- More general view on the FMM
- Theory of Diagonal Forms of Translation Operators in 1D and 2D (Laplace)
  - Renormalized Basis Functions
  - Toeplitz and Hankel Translation Matrices
  - Signature Functions for R and S bases
  - Integral Representations
  - Translation Operators
  - FMM with Diagonal Translations
  - Example
  - More Insight to Diagonal Forms

More General View on the FMM

Continuous Function → (Exact) → Representation

Representation ↓ (Error) ↓ Finite Representation

Representation Errors

Translation Errors

Evaluation Errors

FMM:
- S-Representation
- S|S-Translations
- S|R-Translations
- R|R-Translations
- R-Evaluation

More General View on the FMM (2)

FMM:	
S-Representation	→ Build Finite Information on Function
S|S-Translations	→ Translate Information to Coarser Level
S|R-Translations	→ Convert Information
R|R-Translations	→ Translate Information to Finer Level
R-Evaluation	→ Convert Information to Function Value

In any case: ``Information" is a vector of finite length.
Translation/Conversion means transform of this vector.
If representations are linear, they are represented by some matrices acting on the vector.

Renormalized R-functions

$$\widetilde{R}_n(y) = \frac{y^n}{n!}.$$

Then

$$\widetilde{R}_n(y+t) = \frac{1}{n!}(y+t)^n = \frac{1}{n!}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}t^{n-m}y^m = \sum_{m=0}^{n}\widetilde{R}_{n-m}(t)\widetilde{R}_m(y).$$

$$\left(\widetilde{R}|\widetilde{R}\right)_{mn}(t) = \begin{cases} 0, & m > n \\ \frac{1}{(n-m)!}t^{n-m} = \widetilde{R}_{n-m}(t), & m \leqslant n \end{cases}.$$

Translation Matrix:

$$\left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t) = \left(\widetilde{R}|\widetilde{R}\right)_{mn}(t) = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \dots \\ 0 & 1 & t & \frac{t^2}{2} & \dots \\ 0 & 0 & 1 & t & \dots \\ 0 & 0 & 0 & 1 & \dots \end{pmatrix}.$$

Toeplitz

Renormalized S-functions

$$\widetilde{S}_n(y) = \frac{(-1)^n n!}{y^{n+1}}.$$

$$\widetilde{S}_n(y+t) = (-1)^n n!(y+t)^{-n-1} = (-1)^n n! \sum_{m=n}^{\infty} \frac{(-1)^{m-n} m!}{n!(m-n)!} t^{m-n} y^{-m-1} = \sum_{m=n}^{\infty} \widetilde{R}_{m-n}(t) \widetilde{S}_m(y).$$

$$\left(\widetilde{S}|\widetilde{S}\right)_{mn}(t) = \begin{cases} 0, & m < n \\ \frac{1}{(m-n)!} t^{m-n} = \widetilde{R}_{m-n}(t), & m \geqslant n \end{cases}.$$

Translation Matrix:

Toeplitz

$$\left(\widetilde{\mathbf{S}}|\widetilde{\mathbf{S}}\right)(t) = \left(\widetilde{S}|\widetilde{S}\right)_{mn}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ t & 1 & 0 & 0 & \ldots \\ \frac{t^2}{2} & t & 1 & 0 & \ldots \\ \frac{t^3}{6} & \frac{t^2}{2} & t & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} = \left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)^T(t).$$

Renormalized S-functions

$$\widetilde{S}_n(y+t) = \sum_{m=n}^{\infty} \widetilde{R}_{m-n}(y)\widetilde{S}_m(t) = \sum_{m=0}^{\infty} \widetilde{S}_{m+n}(t)\widetilde{R}_m(y).$$

$$\left(\widetilde{S}|\widetilde{R}\right)_{mn}(t) = \widetilde{S}_{m+n}(t).$$

Hankel

Translation Matrix:

$$\left(\widetilde{\mathbf{S}}|\widetilde{\mathbf{R}}\right)(t) = \left(\widetilde{S}|\widetilde{R}\right)_{mn}(t) = \begin{pmatrix} t^{-1} & -t^{-2} & 2t^{-3} & -6t^{-4} & \dots \\ -t^{-2} & 2t^{-3} & -6t^{-4} & 24t^{-5} & \dots \\ 2t^{-3} & -6t^{-4} & 24t^{-5} & -120t^{-6} & \dots \\ -6t^{-4} & 24t^{-5} & -120t^{-6} & 720t^{-7} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Toeplitz Matrices

$$W_{mn} = U_{m-n}, \quad m = 0, \ldots, M-1, \quad n = 0, \ldots, N-1.$$

A Toeplitz matrix has the following structure:

$$\mathbf{W} = \{W_{mn}\} = \begin{pmatrix} W_{00} & W_{01} & \ldots & W_{0,N-1} \\ W_{10} & W_{11} & \ldots & W_{1,N-1} \\ \ldots & \ldots & \ldots & \ldots \\ W_{M-1,0} & W_{M-1,1} & \ldots & W_{M-1,N-1} \end{pmatrix} = \begin{pmatrix} U_0 & U_{-1} & \ldots & U_{-N+1} \\ U_1 & U_0 & \ldots & U_{1-N} \\ \ldots & \ldots & \ldots & \ldots \\ U_{M-1} & U_{M-1} & \ldots & U_{M-N} \end{pmatrix}.$$

Hankel Matrices

$$W_{mn} = U_{m+n}, \quad m = 0, \ldots, M-1, \quad n = 0, \ldots, N-1.$$

So the structure of the Hankel matrix is

$$\mathbf{W} = \{W_{mn}\} = \begin{pmatrix} W_{00} & W_{01} & \cdots & W_{0,N-1} \\ W_{10} & W_{11} & \cdots & W_{1,N-1} \\ \cdots & \cdots & \cdots & \cdots \\ W_{M-1,0} & W_{M-1,1} & \cdots & W_{M-1,N-1} \end{pmatrix}$$

$$= \begin{pmatrix} U_0 & U_1 & \cdots & U_{N-1} \\ U_1 & U_2 & \cdots & U_N \\ \cdots & \cdots & \cdots & \cdots \\ U_{M-1} & U_M & \cdots & U_{N+M-1} \end{pmatrix},$$

Fast Toeplitz Matrix-Vector Product

$$B_m = \sum_{n=0}^{N-1} W_{mn} A_n = \sum_{n=0}^{N-1} U_{m-n} A_n, \quad m = 0, \ldots, M-1.$$

$$u(t) = \sum_{l=-N+1}^{M-1} U_l e^{ilt}, \quad b(t) = \sum_{m=0}^{M-1} B_m e^{imt}, \quad a(t) = \sum_{n=0}^{N-1} A_n e^{int}.$$

$$a(t)u(t) = \sum_{l=-N+1}^{M-1} \sum_{n=0}^{N-1} A_n U_l e^{i(n+l)t} = \sum_{m=-N+1}^{M+N-2} e^{imt} \sum_{n=0}^{N-1} A_n U_{m-n}$$

$$= \sum_{m=-N+1}^{-1} \alpha_m e^{imt} + b(t) + \sum_{m=M}^{M+N-2} \beta_m e^{imt}.$$

Algorithm:

1). Perform the inverse Fourier transforms of U and A, to determine $u(t)$ and $a(t)$ sampled at appropriate points $\{t_j\}$ (operation of complexity $O((N+M)\log(N+M))$);

2). Multiply point-by-point two vectors $\{u(t_j)\}$ and $\{a(t_j)\}$ to get $\{b(t_j)\}$ (complexity $O(N+M)$);

3). Perform the forward Fourier transform of $\{b(t_j)\}$ (complexity $O((N+M)\log(N+M))$), and take coefficients of harmonics from $m = 0$ to $m = M-1$, which are B_m.

Fast Hankel Matrix-Vector Product

$$B_m = \sum_{n=0}^{N-1} W_{mn} A_n = \sum_{n=0}^{N-1} U_{m+n} A_n, \quad m = 0,\ldots,M-1.$$

$$u(t) = \sum_{l=0}^{M+N-2} U_l e^{ilt}, \quad b(t) = \sum_{m=0}^{M-1} B_m e^{imt}, \quad a(t) = \sum_{n=0}^{N-1} \overline{A_n} e^{int}.$$

$$\overline{a(t)}u(t) = \sum_{l=0}^{M+N-2} \sum_{n=0}^{N-1} A_n U_l e^{i(l-n)t} = \sum_{m=-N+1}^{M+N-2} e^{imt} \sum_{n=0}^{N-1} A_n U_{m+n}$$

$$= \sum_{m=-N+1}^{-1} \alpha_m e^{imt} + b(t) + \sum_{m=M}^{M+N-2} \beta_m e^{imt}.$$

Algorithm:

1). Perform the inverse Fourier transforms of $\mathbf{U}$ and $\overline{\mathbf{A}}$, to determine $u(t)$ and $a(t)$ sampled at appropriate points $\{t_j\}$ (operation of complexity $O((N+M)\log(N+M))$);

2). Multiply point-by-point two vectors $\{u(t_j)\}$ and $\left\{\overline{a(t_j)}\right\}$ to get $\{b(t_j)\}$ (complexity $O(N+M)$);

3). Perform the forward Fourier transform of $\{b(t_j)\}$ (complexity $O((N+M)\log(N+M))$), and take coefficients of harmonics from $m = 0$ to $m = M-1$, which are B_m.

With such renormalized functions all translations can be performed with complexity O($p\log p$).

But we look for something faster.

Theoretical limit for translation of vector of length p is $O(p)$.

ONLY SPARSE TRANSLATION MATRIX CAN PROVIDE SUCH COMPLEXITY

Representations Based on Signature Functions

Definition *Let*

$$\Phi(y) = \sum_{m=0}^{\infty} C_m \widetilde{R}_m(y),$$

then the Signature Function of $\Phi(y)$ *is a* 2π*-periodic funcion*

$$\Phi^*(s) = \sum_{m=0}^{\infty} C_m e^{ims}.$$

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Definition *Let*

$$\Phi(y) = \sum_{m=0}^{\infty} C_m \widetilde{S}_m(y),$$

then the Signature Function of $\Phi(y)$ *is a* 2π*-periodic funcion*

$$\Phi^*(s) = \sum_{m=0}^{\infty} C_m e^{-ims}.$$

We assume that series for SF converge. This is always true for finite series, $C_m = 0,\ m > p\text{-}1$.

Integral Representation of Regular Functions

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} \Phi^*(s) e^{-ims} ds.$$ ← Property of Fourier coefficients

We have then the following representation of $\Phi(y)$:

$$\Phi(y) = \sum_{m=0}^{\infty} \tilde{R}_m(y) \frac{1}{2\pi}\int_0^{2\pi} \Phi^*(s) e^{-ims} ds = \frac{1}{2\pi}\int_0^{2\pi} \Phi^*(s) \sum_{m=0}^{\infty} \tilde{R}_m(y) e^{-ims} ds$$

Consider

$$\sum_{m=0}^{\infty} \tilde{R}_m(y) e^{-ims} = \sum_{m=0}^{\infty} e^{-ims}\frac{y^m}{m!} = \sum_{m=0}^{\infty} \frac{\left(ye^{-is}\right)^m}{m!} = e^{ye^{-is}}.$$

So

$$\Phi(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}} \Phi^*(s) ds = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y;s)\Phi^*(s) ds,$$

where

$$\Lambda_r(y;s) = e^{ye^{-is}}.$$ ← Regular kernel

Integral Representation of Regular Basis Functions

For $\Phi(y) = \widetilde{R}_m(y)$ we have

$$\Phi(y) = \widetilde{R}_m(y) = \sum_{m'=0}^{\infty} C_{m'} \widetilde{R}_{m'}(y), \qquad C_{m'} = \delta_{mm'}$$

Therefore the SF for this function is

$$\Phi^*(s) = \sum_{m'=0}^{\infty} C_{m'} e^{im's} = \sum_{m'=0}^{\infty} \delta_{mm'} e^{im's} = e^{ims}.$$

Then

$$\widetilde{R}_m(y) = \Phi(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}} \Phi^*(s)ds = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}} e^{ims}ds = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y;s)e^{ims}ds.$$

Integral Representation of Singular Functions

$$\Phi^{(p)}(y) = \sum_{m=0}^{p-1} C_m \widetilde{S}_m(y), \qquad \Phi^{(p)*}(s) = \sum_{m=0}^{p-1} C_m e^{-ims}.$$

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{(p)*}(s) e^{ims} ds.$$

← Property of Fourier coefficients

We have then the following representation of $\Phi(y)$:

$$\Phi^{(p)}(y) = \sum_{m=0}^{p-1} \widetilde{S}_m(y) \frac{1}{2\pi}\int_0^{2\pi} \Phi^{(p)*}(s) e^{ims} ds = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{(p)*}(s) \sum_{m=0}^{p-1} \widetilde{S}_m(y) e^{ims} ds$$

Then

$$\Phi^{(p)}(y) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_s^{(p)}(y; s) \Phi^{(p)*}(s) ds,$$

$$\Lambda_s^{(p)}(y; s) = \sum_{m=0}^{p-1} \widetilde{S}_m(y) e^{ims} = \sum_{m=0}^{p-1} e^{ims} \frac{(-1)^m m!}{y^{m+1}}.$$

← Singular kernel

Integral Representation of Singular Basis Functions

For $\Phi(y) = \widetilde{S}_m(y)$ we have

$$\Phi^{(p)}(y) = \widetilde{S}_m(y) = \sum_{m'=0}^{p-1} C_{m'} \widetilde{S}_{m'}(y), \qquad C_{m'} = \delta_{mm'}, \qquad p > m.$$

Therefore the SF for this function is

$$\Phi^{(p)*}(s) = \sum_{m'=0}^{\infty} C_{m'} e^{-im's} = \sum_{m'=0}^{\infty} \delta_{mm'} e^{-im's} = e^{-ims}, \qquad p > m.$$

Then

$$\widetilde{S}_m(y) = \Phi^{(p)}(y) = \frac{1}{2\pi} \int_0^{2\pi} \Lambda_s^{(p)}(y;s) \Phi^{(p)*}(s) ds = \frac{1}{2\pi} \int_0^{2\pi} \Lambda_s^{(p)}(y;s) e^{-ims} ds,$$

$$m < p.$$

R|R-translation of the Signature Function

$$\mathcal{T}(t)[\Phi(y)] = \Phi(y+t) = \frac{1}{2\pi}\int_0^{2\pi} e^{(y+t)e^{-is}}\Phi^*(s)ds = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}}e^{te^{-is}}\Phi^*(s)ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Lambda_r(y,s)\Lambda_r(t,s)\Phi^*(s)ds = \frac{1}{2\pi}\int_0^{2\pi}\Lambda_r(y,s)\widehat{\Phi}^*(s,t)ds.$$

$$(\mathcal{R}|\mathcal{R})(t)[\Phi^*(s)] = \widehat{\Phi}^*(s,t) = \Lambda_r(t,s)\Phi^*(s).$$

So the R|R translation of the SF means simply multiplication of the SF by the regular kernel !

S|S-translation of the Signature Function

$$\Phi^{(p)}(y+t) = \sum_{m=0}^{p-1} \widehat{C}_m \widetilde{S}_m(y) = \sum_{m=0}^{p-1}\sum_{n=0}^{\infty} \left(\widetilde{S}|\widetilde{S}\right)_{mn}(t) C_n \widetilde{S}_m(y)$$

$$= \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \widetilde{R}_{m-n}(t) C_n \widetilde{S}_m(y) = \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \frac{1}{2\pi}\int_0^{2\pi} e^{te^{-is}} e^{i(m-n)s} ds\, C_n \widetilde{S}_m(y)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{te^{-is}} \sum_{n=0}^{p-1} C_n e^{-ins} \sum_{m=0}^{p-1} \widetilde{S}_m(y) e^{ims} ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Lambda_s^{(p)}(y,s) e^{te^{-is}} \Phi^{(p)*}(s) ds, \qquad |t| < |y|.$$

Representation of the regular basis function

So

$$(\mathcal{S}|\mathcal{S})(t)[\Phi^{(p)*}(s)] = \widehat{\Phi}^{(p)*}(s;t) = e^{te^{-is}} \Phi^{(p)*}(s) = \Lambda_r(t,s)\Phi^{(p)*}(s).$$

So the S|S translation of the SF means multiplication of the SF by the regular kernel.

S|R-translation of the Signature Function

In case $|t| > |y|$ we have

$$\Phi^{(p)}(y+t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}} \Lambda_s^{(p)}(t,s)\Phi^{(p)*}(s)ds, \qquad |t| > |y|.$$

This is a representation of the regular function. Therefore,

$$(\mathcal{S}|\mathcal{R})(t)[\Phi^{(p)*}(s)] = \widehat{\Phi}^{(p)*}(s;t) = \Lambda_s^{(p)}(t,s)\Phi^{(p)*}(s).$$

So the S|S translation of the SF means multiplication of the SF by the singular kernel.

Evaluation of Function based on its Signature Function

Use Gaussian Type Quadrature

function bandwidth

$$\Phi^{(p)}(y) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y;s)\Phi^{(p)*}(s)ds$$

$$= \sum_{k=0}^{q-1} w_k \Lambda_r(y;s_k)\Phi^{(p)*}(s_k) + error(p,q).$$

quadrature weights

quadrature nodes

quadrature order

This Provides the MLFMM based on SF translations. To link it with the MLFMM in space of S- and R- expansion coefficients consider the following correspondence.

Correspondence between samples of the SF and expansion coefficients

Let

$$\Phi^{(p)}(y) = \sum_{m=0}^{p-1} C_m \widetilde{R}_m(y), \qquad \Phi^{(p)*}(s) = \sum_{m=0}^{p-1} C_m e^{ims}$$

is an acceptable p-truncated approximation. The SF can be defined by p samples in an equispaced grid:

$$s_k = \frac{2\pi k}{p}, \qquad k = 0, \ldots, p-1.$$

Then

Inverse FFT of order p

$$\Phi_k^{(p)*} = \Phi^{(p)*}(s_k) = \sum_{m=0}^{p-1} C_m e^{ims_k}, \qquad \left\{\Phi_k^{(p)*}\right\} = IFFT_{(p)}\{C_m\}.$$

$$C_m = \frac{1}{p}\sum_{m=0}^{p-1} \Phi^{(p)*}(s_k) e^{-ims_k}, \qquad \{C_m\} = FFT_{(p)}\left\{\Phi_k^{(p)*}\right\}.$$

So we have one-to-one correspondence

$$\{C_m\} \rightleftarrows \left\{\Phi_k^{(p)*}\right\}$$

Chain of translations in MLFMM

$$\left\{C_m^{(l+1)}\right\} = \left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t_l)\left\{C_m^{(l)}\right\}$$

$$\left\{\Phi_k^{*(l)}\right\} = IFFT\left\{C_m^{(l)}\right\} = \mathbf{F}\left\{C_m^{(l)}\right\}$$

$$\left\{\Phi_k^{*(l+1)}\right\} = (\mathcal{R}|\mathcal{R})(t_l)\left\{\Phi_k^{*(l)}\right\} \qquad \left(\Phi_k^{*(l+1)} = \Lambda_r(t_{l+1,l}, s_k)\Phi_k^{*(l)}\right)$$

$$\left\{C_m^{(l+1)}\right\} = FFT\left\{\Phi_k^{*(l+1)}\right\} = \mathbf{F}^{-1}\left\{\Phi_k^{*(l+1)}\right\}$$

$$\left\{C_m^{(l+1)}\right\} = \left[\mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(t_l)\mathbf{F}\right]\left\{C_m^{(l)}\right\}$$

Chain:

$$\left\{C_m^{(L)}\right\} = \left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t_{L-1})\left\{C_m^{(L-1)}\right\} = \left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t_{L-1})\left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t_{L-2})\left\{C_m^{(L-2)}\right\}$$

$$= \left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t_{L-1})\left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t_{L-2})\ldots\left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t_l)\left\{C_m^{(l)}\right\}$$

$$\left\{C_m^{(L)}\right\} = \mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(t_{L-1})\mathbf{F}\mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(t_{L-2})\mathbf{F}^{-1}\ldots\mathbf{F}(\mathcal{R}|\mathcal{R})(t_l)\mathbf{F}^{-1}\left\{C_m^{(l)}\right\}$$

$$= \mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(t_{L-1})(\mathcal{R}|\mathcal{R})(t_{L-2})\ldots(\mathcal{R}|\mathcal{R})(t_l)\mathbf{F}\left\{C_m^{(l)}\right\}$$

$$\left\{\Phi_k^{*(L)}\right\} = (\mathcal{R}|\mathcal{R})(t_{L-1})(\mathcal{R}|\mathcal{R})(t_{L-2})\ldots(\mathcal{R}|\mathcal{R})(t_l)\left\{\Phi_k^{*(l)}\right\}$$

Translations of SF:

$$\left\{C_m^{(L)}\right\} = \mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(t_{L-1})\mathbf{F}\mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(t_{L-2})\mathbf{F}^{-1}...\mathbf{F}(\mathcal{R}|\mathcal{R})(t_l)\mathbf{F}^{-1}\left\{C_m^{(l)}\right\}$$

$$= \mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(t_{L-1})(\mathcal{R}|\mathcal{R})(t_{L-2})...(\mathcal{R}|\mathcal{R})(t_l)\mathbf{F}\left\{C_m^{(l)}\right\}$$

$$\left\{\Phi_k^{*(L)}\right\} = (\mathcal{R}|\mathcal{R})(t_{L-1})(\mathcal{R}|\mathcal{R})(t_{L-2})...(\mathcal{R}|\mathcal{R})(t_l)\left\{\Phi_k^{*(l)}\right\}$$

This can be done for each s_k independently!
Since at the final step only integration is required
s_k are just quadrature nodes!

MLFMM using SF representation

Upward Pass:

Step 1: Build SF for singular expansion sampled at p points for each (nth) source

$$\Phi_n^{(p)*}(s_k) = u_n \sum_{m=0}^{p-1} C_m e^{-ims_k}, \qquad k = 0,...,p-1, \qquad n = 1,...,N$$

Step 2: Perform regular FMM procedure where the S|S-operator is

$$(\mathcal{S}|\mathcal{S})(t)[\Phi^{(p)*}(s_k)] = \Lambda_r(t;s_k)\Phi^{(p)*}(s_k), \qquad k = 0,...,p-1,$$

Downward Pass:

Step 1: Perform regular FMM procedure where the S|R-operator is

$$(\mathcal{S}|\mathcal{R})(t)[\Phi^{(p)*}(s_k)] = \Lambda_s^{(p)}(t;s_k)\Phi^{(p)*}(s_k), \qquad k = 0,...,p-1,$$

Step 2: Perform regular FMM procedure where the R|R-operator is

$$(\mathcal{R}|\mathcal{R})(t)[\Phi^{(p)*}(s_k)] = \Lambda_r(t;s_k)\Phi^{(p)*}(s_k), \qquad k = 0,...,p-1.$$

Final Summation:

Evaluate the potential. The near field is computed as usual while the contribution of the far field is

$$\Phi^{(p)}(y_j) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y_j;s)\Phi^{(p)*}(s)ds = \sum_{k=0}^{p-1} w_k \Lambda_r(y_j;s_k)\Phi^{(p)*}(s_k), \qquad j = 1,...,M.$$

Example

$$v(y) = \sum_{n=1}^{N} \Phi_n(y, x_n), \qquad \Phi_n(y, x_l) = \frac{u_n}{y - x_n}.$$

S-expansion:

$$\Phi_n(y, x_n) = \frac{u_n}{y - x_n} = u_n \sum_{m=0}^{\infty} b_m(x_n, x_*) S_m(y - x_*) = u_n \sum_{m=0}^{\infty} (x_n - x_*)^m (y - x_*)^{-m-1}$$

$$= u_n \sum_{m=0}^{\infty} (-1)^m \frac{(x_n - x_*)^m}{m!} \left[(-1)^m m! (y - x_*)^{-m-1} \right]$$

$$= u_n \sum_{m=0}^{\infty} (-1)^m \widetilde{R}_m(x_n - x_*) \widetilde{S}_m(y - x_*) = u_n \sum_{m=0}^{\infty} \widetilde{R}_m(x_* - x_n) \widetilde{S}_m(y - x_*).$$

So here

$$C_m = \widetilde{R}_m(x_* - x_n).$$

Example (2)

$$\Phi_n^*(s) = u_n \sum_{m=0}^{\infty} C_m e^{-ims} = u_n \sum_{m=0}^{\infty} \widetilde{R}_m(x_* - x_n) e^{-ims}$$

$$= u_n \sum_{m=0}^{\infty} \frac{(x_* - x_n)^m}{m!} e^{-ims} = u_n \sum_{m=0}^{\infty} \frac{\left[e^{-is}(x_* - x_n)\right]^m}{m!}$$

$$= u_n e^{(x_* - x_n)e^{-is}} = u_n \Lambda_r(x_* - x_n; s).$$

Therefore, the SF due to sources in box with center x_* is

$$\Phi^*(s) = \sum_{x_n \in box(x_*)} \Phi_n^*(s) = \sum_{x_n \in box(x_*)} u_n \Lambda_r(x_* - x_n; s).$$

Asymptotic complexity of the MLFMM in this case is $O(p(N+M))$. The same as for Middleman!

More Insight to Diagonal Forms

Theorem: for any λ vector $(1, \lambda, \lambda^2, \ldots)$ is an eigen vector of matrix $\left(\widetilde{\mathbf{R}} | \widetilde{\mathbf{R}}\right)(t)$ and corresponds to eigenvalue $e^{\lambda t}$.

Proof:

$$\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \ldots \\ 0 & 1 & t & \frac{t^2}{2} & \ldots \\ 0 & 0 & 1 & t & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \ldots \end{pmatrix} = \begin{pmatrix} 1 + \lambda t + \frac{\lambda^2 t^2}{2} + \ldots \\ \lambda\left(1 + \lambda t + \frac{\lambda^2 t^2}{2} + \ldots\right) \\ \lambda^2\left(1 + \lambda t + \frac{\lambda^2 t^2}{2} + \ldots\right) \\ \lambda^3\left(1 + \lambda t + \frac{\lambda^2 t^2}{2} + \ldots\right) \\ \ldots \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \ldots \end{pmatrix}.$$

More Insight to Diagonal Forms (2)

This means that the spectrum of the $R|R$-translation operator is continuous. We approximate this spectrum with finite spectrum, $\lambda_k = e^{-is_k}$, $k = 0, ..., p-1$. So the k-th eigen vector is a column of the inverse Fourier matrix $\mathbf{F}^{-1}$

$$\mathbf{v}_k = \begin{pmatrix} 1 \\ e^{-is_k} \\ e^{-2is_k} \\ ... \\ e^{-(p-1)is_k} \end{pmatrix}, \qquad \mathbf{F}^{-1} = \frac{1}{p}\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ e^{-is_0} & e^{-is_1} & e^{-is_2} & ... & e^{-is_{p-1}} \\ e^{-2is_0} & e^{-2is_1} & e^{-2is_2} & ... & e^{-2is_{p-1}} \\ ... & ... & ... & ... & ... \\ e^{-(p-1)is_0} & e^{-(p-1)is_1} & e^{-(p-1)is_2} & ... & e^{-(p-1)is_{p-1}} \end{pmatrix}$$

Then we have a decomposition consistent with what we found above:

$$\left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t) = \mathbf{F}^{-1}\begin{pmatrix} e^{te^{-is_0}} & 0 & 0 & ... & 0 \\ 0 & e^{te^{-is_1}} & 0 & ... & 0 \\ 0 & 0 & e^{te^{-is_2}} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & e^{te^{-is_{p-1}}} \end{pmatrix}\mathbf{F} = \mathbf{F}^{-1}(\mathcal{R}|\mathcal{R})(\mathbf{t})\mathbf{F}.$$

Problems with the SF-method

- Each multiplication of the signature function by the translation kernel increases the bandwidth of the signature function;
- Filtering procedures are needed to control the error and keep the prescribed bandwidth;
- Filtering procedures can be done with the FFT. In this case the complexity of translation with filtering will be $O(p \log p)$ (in fact, this is the same as Toeplitz or Hankel Matrix multiplication).