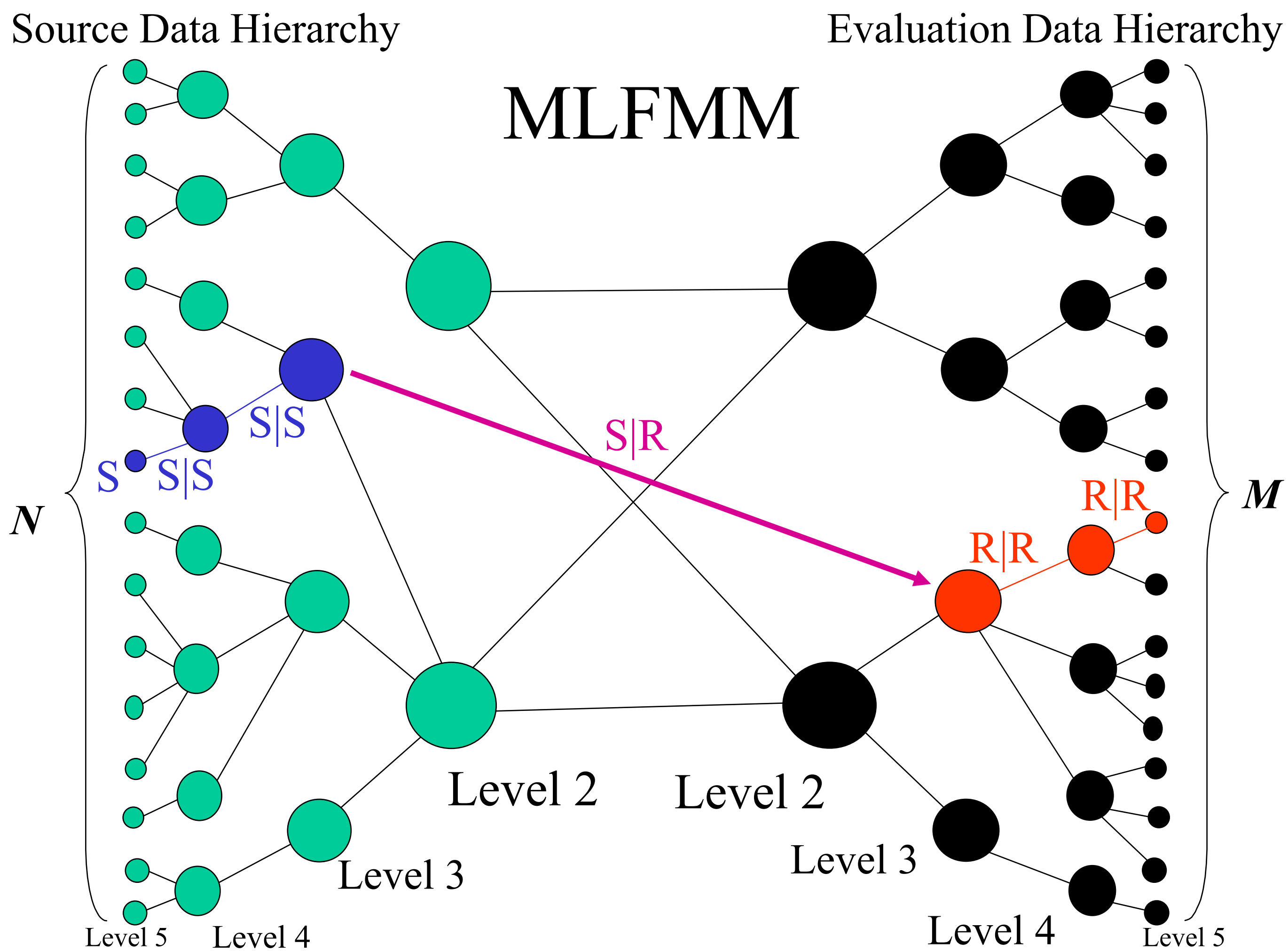
FMM CMSC 878R/AMSC 698R

Lecture 14

Outline

- Error Bounds of MLFMM
  - A scheme for error evaluation;
- Example problem
  - S-expansion error;
  - S|S-translation error;
  - S|R-translation error;
  - R|R-translation error.
- Error and Neighborhoods

A scheme for error evaluation (1)
(How one source contributes to one evaluation points)



A scheme for error evaluation (2)

$$\Phi(\mathbf{y},\mathbf{x}_k) = \sum_{m=0}^{\infty} C_m^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)}) S_m(\mathbf{y}-\mathbf{x}_*^{(L)}) = \mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L)})$$

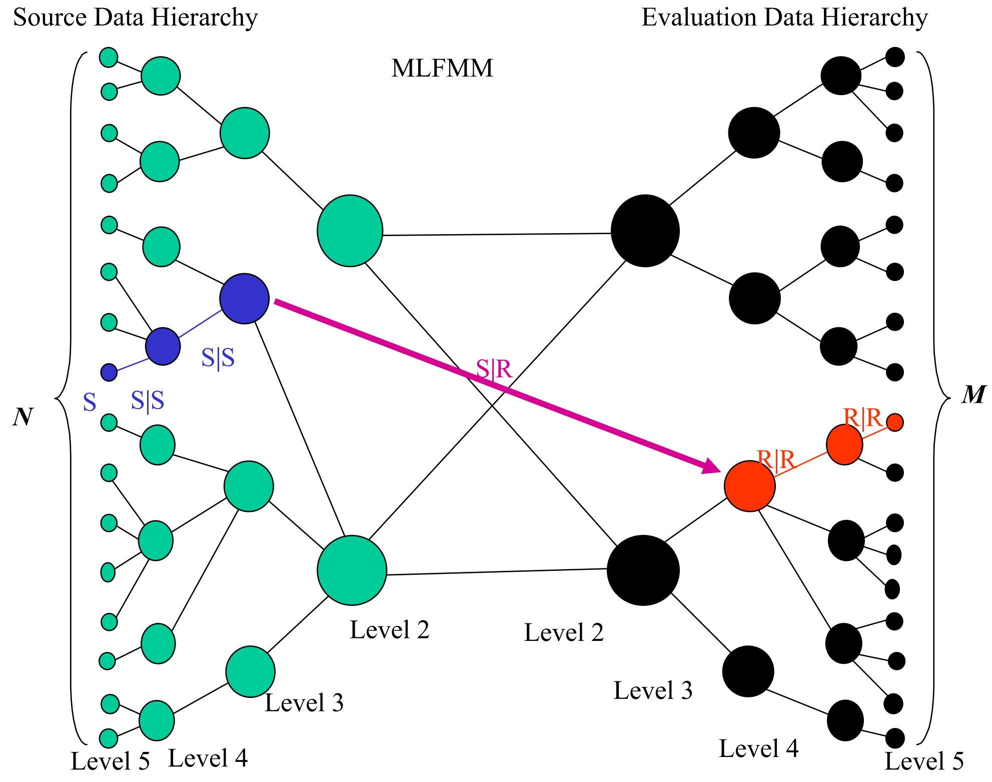
$$= \mathbf{C}^{(L-1)}(\mathbf{x}_k,\mathbf{x}_*^{(L-1)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L-1)})$$

$$= \ldots = \mathbf{C}^{(l)}(\mathbf{x}_k,\mathbf{x}_*^{(l)}) \cdot \mathbf{S}\left(\mathbf{y}-\mathbf{x}_*^{(l)}\right)$$

$$= \mathbf{D}^{(l)}(\mathbf{x}_k,\mathbf{y}_*^{(l)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l)}\right)$$

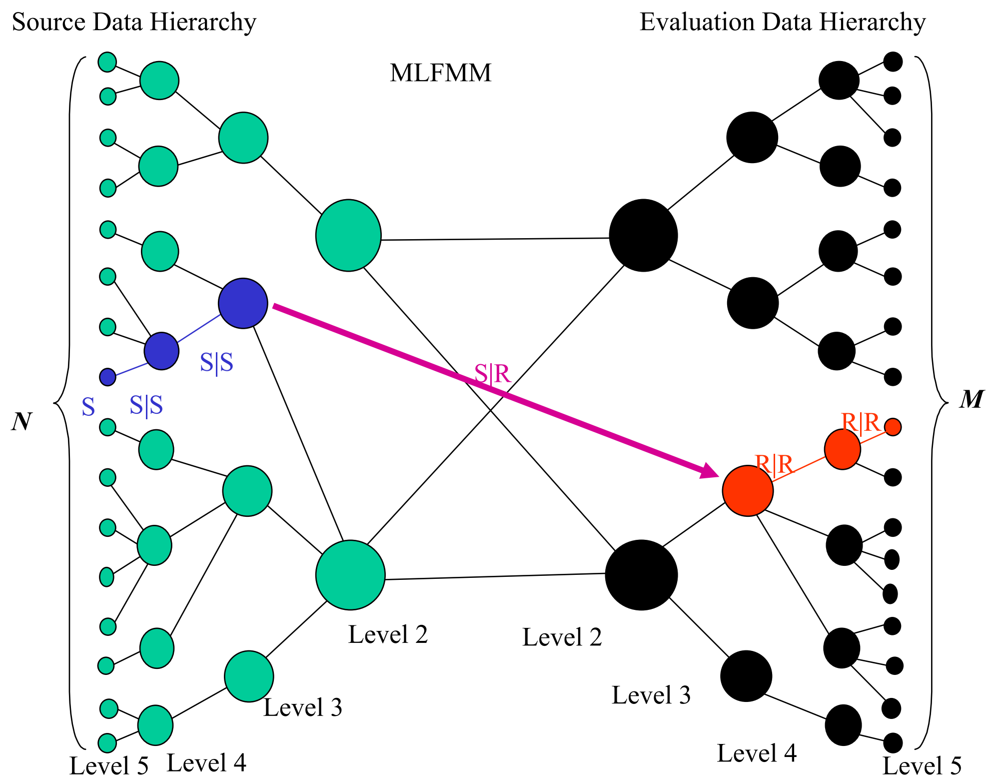
$$= \mathbf{D}^{(l+1)}(\mathbf{x}_k,\mathbf{y}_*^{(l+1)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l+1)}\right)$$

$$= \ldots = \mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) \cdot \mathbf{R}(\mathbf{y}-\mathbf{y}_*^{(L)}).$$



A scheme for error evaluation (3)

$$
\begin{aligned}
\mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) &= (\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\mathbf{D}^{(L-1)}(\mathbf{x}_k,\mathbf{y}_*^{(L-1)}) \\
&= [(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L-1)}-\mathbf{y}_*^{(L-2)})]\mathbf{D}^{(L-2)}(\mathbf{x}_k,\mathbf{y}_*^{(L-2)}) \\
&= \ldots = \left[(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L-1)}-\mathbf{y}_*^{(L-2)})\circ\ldots\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\right]\mathbf{D}^{(l)}(\mathbf{x}_k,\mathbf{y}_*^{(l)}) \\
&= \left[(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\ldots\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\circ(\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right)\right]\mathbf{C}^{(l)}(\mathbf{x}_k,\mathbf{x}_*^{(l)}) \\
&= \left[(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\ldots\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\circ(\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right)\circ(\mathbf{S}|\mathbf{S})\left(\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right)\right]\mathbf{C}^{(l+1)}(\mathbf{x}_k,\mathbf{x}_*^{(l+1)}) \\
&= \ldots \\
&= \left[(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\ldots\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\circ(\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right)\circ(\mathbf{S}|\mathbf{S})\left(\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right)\circ\ldots\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-1)}-\mathbf{x}_*^{(L)})\right]\mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)})
\end{aligned}
$$



A scheme for error evaluation (4)

Consider computation of the final coefficients with p-truncated matrices

$$\mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) = [\Pr(p)\circ(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})\circ\Pr(p)]\circ\ldots\circ$$
$$\left[\Pr(p)\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\circ\Pr(p)\right]\circ$$
$$\left[\Pr(p)\circ(\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right)\circ\Pr(p)\right]\circ$$
$$\left[\Pr(p)\circ(\mathbf{S}|\mathbf{S})\left(\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right)\circ\Pr(p)\right]\circ\ldots\circ$$
$$[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-2)}-\mathbf{x}_*^{(L-1)})\circ\Pr(p)]\circ$$
$$[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-1)}-\mathbf{x}_*^{(L)})\circ\Pr(p)]\mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)})$$

These truncation operators can be skipped! ($\Pr^2=\Pr$)

So:

$$\mathbf{D}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) = [\Pr(p)\circ(\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(L)}-\mathbf{y}_*^{(L-1)})]\circ\ldots\circ$$
$$\left[\Pr(p)\circ(\mathbf{R}|\mathbf{R})\left(\mathbf{y}_*^{(l+1)}-\mathbf{y}_*^{(l)}\right)\right]\circ$$
$$\left[\Pr(p)\circ(\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)}-\mathbf{x}_*^{(l)}\right)\right]\circ$$
$$\left[\Pr(p)\circ(\mathbf{S}|\mathbf{S})\left(\mathbf{x}_*^{(l)}-\mathbf{x}_*^{(l+1)}\right)\right]\circ\ldots\circ$$
$$[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-2)}-\mathbf{x}_*^{(L-1)})]\circ$$
$$[\Pr(p)\circ(\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(L-1)}-\mathbf{x}_*^{(L)})]\circ\Pr(p)\mathbf{C}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)})$$

A scheme for error evaluation (5)

p-truncated functions:

$$\hat{\Phi}_L(\mathbf{y},\mathbf{x}_k) = \hat{\mathbf{C}}^{(L)}(\mathbf{x}_k,\mathbf{x}_*^{(L)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L)})$$

$$\hat{\Phi}_{L-1}(\mathbf{y},\mathbf{x}_k) = \hat{\mathbf{C}}^{(L-1)}(\mathbf{x}_k,\mathbf{x}_*^{(L-1)}) \cdot \mathbf{S}(\mathbf{y}-\mathbf{x}_*^{(L-1)})$$

$$\ldots$$

$$\hat{\Phi}_l(\mathbf{y},\mathbf{x}_k) = \hat{\mathbf{C}}^{(l)}(\mathbf{x}_k,\mathbf{x}_*^{(l)}) \cdot \mathbf{S}\left(\mathbf{y}-\mathbf{x}_*^{(l)}\right)$$

$$\hat{\Psi}_l(\mathbf{y},\mathbf{x}_k) = \hat{\mathbf{D}}^{(l)}(\mathbf{x}_k,\mathbf{y}_*^{(l)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l)}\right)$$

$$\hat{\Psi}_{l+1}(\mathbf{y},\mathbf{x}_k) = \hat{\mathbf{D}}^{(l+1)}(\mathbf{x}_k,\mathbf{y}_*^{(l+1)}) \cdot \mathbf{R}\left(\mathbf{y}-\mathbf{y}_*^{(l+1)}\right)$$

$$\ldots$$

$$\hat{\Psi}_L(\mathbf{y},\mathbf{x}_k) = \hat{\mathbf{D}}^{(L)}(\mathbf{x}_k,\mathbf{y}_*^{(L)}) \cdot \mathbf{R}(\mathbf{y}-\mathbf{y}_*^{(L)}).$$

The error comes only from truncation operator

$$\hat{\mathbf{C}}^{(\alpha)} = \Pr(p) \circ (\mathbf{S}|\mathbf{S})(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)})\hat{\mathbf{C}}^{(\alpha+1)}, \quad \alpha = L-1,\ldots,l$$

$$\hat{\mathbf{D}}^{(l)} = \Pr(p) \circ (\mathbf{S}|\mathbf{R})\left(\mathbf{y}_*^{(l)} - \mathbf{x}_*^{(l)}\right)\hat{\mathbf{C}}^{(l)},$$

$$\hat{\mathbf{D}}^{(\alpha)} = \Pr(p) \circ (\mathbf{R}|\mathbf{R})(\mathbf{y}_*^{(\alpha)} - \mathbf{y}_*^{(\alpha+1)})\hat{\mathbf{C}}^{(\alpha+1)}, \quad \alpha = l,\ldots,L-1$$

Truncated Translation Theorem

Let $\{F_n(\mathbf{y})\}$ and $\{G_n(\mathbf{y})\}$ be two expansion bases in Ω, and the reexpansion series converges everywhere in Ω :

$$\forall \mathbf{y} \in \Omega, \quad F_n(\mathbf{y}) = \sum_{m=0}^{\infty} (F|G)_{mn} G_m(\mathbf{y}), \quad n = 0, 1, 2, \ldots$$

Let also $\{A_n\}$ be a set of coefficients, such that the duble sum converges absolutely and uniformly in Ω :

$$\forall \mathbf{y} \in \Omega, \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_n (F|G)_{mn} G_m(\mathbf{y}) = \Phi(\mathbf{y}),$$

$$\forall \epsilon, \exists p(\epsilon), \quad \sum_{n=0}^{\infty} \sum_{m=p}^{\infty} |A_n (F|G)_{mn} G_m(\mathbf{y})| < \epsilon, \quad \sum_{n=p}^{\infty} \sum_{m=0}^{\infty} |A_n (F|G)_{mn} G_m(\mathbf{y})| < \epsilon.$$

Then

$$\left| \Phi(\mathbf{y}) - \sum_{n=0}^{p-1} \sum_{m=0}^{p-1} A_n (F|G)_{mn} G_m(\mathbf{y}) \right| < 2\epsilon.$$

Proof

Let us denote

$$c_{mn} = (F|G)_{mn} A_n G_m(\mathbf{y})$$

$$\left|\Phi(\mathbf{y}) - \sum_{n=0}^{p-1}\sum_{m=0}^{p-1} c_{mn}\right| = \left|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{mn} - \sum_{n=0}^{p-1}\sum_{m=0}^{p-1} c_{mn}\right| =$$

$$= \left|\sum_{m=0}^{p-1}\sum_{n=0}^{\infty} c_{mn} + \sum_{m=p}^{\infty}\sum_{n=0}^{\infty} c_{mn} - \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} c_{mn}\right|$$

$$= \left|\sum_{m=0}^{p-1}\sum_{n=0}^{p-1} c_{mn} + \sum_{m=0}^{p-1}\sum_{n=p}^{\infty} c_{mn} + \sum_{m=p}^{\infty}\sum_{n=0}^{\infty} c_{mn} - \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} c_{mn}\right|$$

$$= \left|\sum_{m=0}^{p-1}\sum_{n=p}^{\infty} c_{mn} + \sum_{m=p}^{\infty}\sum_{n=0}^{\infty} c_{mn}\right| \leqslant \sum_{m=0}^{p-1}\sum_{n=p}^{\infty} |c_{mn}| + \sum_{m=p}^{\infty}\sum_{n=0}^{\infty} |c_{mn}|$$

$$\leqslant \sum_{m=0}^{\infty}\sum_{n=p}^{\infty} |c_{mn}| + \sum_{m=p}^{\infty}\sum_{n=0}^{\infty} |c_{mn}| < \epsilon + \epsilon = 2\epsilon.$$

A scheme for error evaluation (5)

For uniformly and absolutely convergent series:

$$\begin{aligned}
&\left|\widehat{\Phi}_{\alpha}(\mathbf{y},\mathbf{x}_k) - \widehat{\Phi}_{\alpha+1}(\mathbf{y},\mathbf{x}_k)\right| \\
&= \left|\sum_{m=0}^{p-1} \widehat{C_m}^{(\alpha)} S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)}) - \sum_{n=0}^{p-1} \widehat{C_n}^{(\alpha+1)} S_n(\mathbf{y}-\mathbf{x}_*^{(\alpha+1)})\right| \\
&= \left|\sum_{m=0}^{p-1}\sum_{n=0}^{p-1} (S|S)_{mn}(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)}) \widehat{C_n}^{(\alpha+1)} S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right. \\
&\quad \left. - \sum_{n=0}^{p-1} \widehat{C_n}^{(\alpha+1)} \sum_{m=0}^{\infty} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| \\
&= \left|\sum_{n=0}^{p-1} \widehat{C_n}^{(\alpha+1)} \sum_{m=p}^{\infty} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| \\
&\leqslant \sum_{n=0}^{p-1}\sum_{m=p}^{\infty} \left|\widehat{C_n}^{(\alpha+1)} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| \\
&< \sum_{n=0}^{\infty}\sum_{m=p}^{\infty} \left|\widehat{C_n}^{(\alpha+1)} (S|S)_{nm}(\mathbf{x}_*^{(\alpha)} - \mathbf{x}_*^{(\alpha+1)}) S_m(\mathbf{y}-\mathbf{x}_*^{(\alpha)})\right| < \epsilon_{\max}(p).
\end{aligned}$$

A scheme for error evaluation (6)

For uniformly and absolutely convergent series it is possible to find such $\varepsilon_{max}(p)$ that for given minimum(maximum) translation distance the max abs difference between two subsequent functions is smaller than $\varepsilon_{max}(p)$.

In this case the total error of FMM does not exceed:

$$FMMError \leqslant N\left[\epsilon_{\max}^{(\exp)}(p) + (L-2)\epsilon_{\max}^{(S|S)}(p) + \epsilon_{\max}^{(S|R)}(p) + (L-2)\epsilon_{\max}^{(R|R)}(p)\right].$$

$$\lim_{p\to\infty}\epsilon_{\max}^{(\exp)}(p) = 0, \quad \lim_{p\to\infty}\epsilon_{\max}^{(S|S)}(p) = 0, \quad \lim_{p\to\infty}\epsilon_{\max}^{(S|R)}(p) = 0, \quad \lim_{p\to\infty}\epsilon_{\max}^{(R|R)}(p) = 0.$$

If

$$\epsilon(p) = \max\left(\epsilon_{\max}^{(\exp)}(p), \epsilon_{\max}^{(S|S)}(p), \epsilon_{\max}^{(S|R)}(p), \epsilon_{\max}^{(R|R)}(p)\right),$$

$$FMMError \leqslant 2N(L-1)\epsilon(p).$$

Different schemes for error estimate are possible.