Problem (Homework 2)

This week's assignment uses the ideas of using Taylor series to achieve factorizations suitable for use in FMM type algorithms. Using this tool to develop a factorization, we will develop a **new version** of the fast Gauss transform (FGT).

Let

$$\Phi_{ji} = e^{-(y_j - x_i)^2}, \quad i = 1, \ldots, N, \quad j = 1, \ldots, M.$$

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \ldots & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & \ldots & \Phi_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ \Phi_{M1} & \Phi_{M2} & \ldots & \Phi_{MN} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \ldots \\ u_N \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \ldots \\ v_M \end{pmatrix}, \qquad \#$$

where $x_1, \ldots, x_N,\ y_1, \ldots, y_M, u_1, \ldots, u_N,$ are random numbers distributed uniformly in $[0, 1]$. Compute the matrix-vector product

$$\mathbf{v} = \mathbf{\Phi u}, \qquad \#$$

or

$$v_j = \sum_{i=1}^{N} \Phi_{ji} u_i, \quad j = 1, \ldots, M, \qquad \#$$

with absolute error $\epsilon < 10^{-6}$. The matrix sizes, $N, M > 0$ are given (fixed) positive integers.

1. Using the example from Lecture #2, write down a factored expression. Estimate the error in truncating the series using residual term evaluation for the Taylor series, and evaluate the truncation number, p, as a function of the required accuracy and N. Provide a formula that can be used for the "fast" ($O(N + M)$) method.
2. Write a program that implements both the straightforward computation based on Eq. (ref: 1.1) and the "fast" method.
3. Plot the absolute maximum error between the straightforward and "Fast" method for $N = 10^3$ and $M = 2N$ and p varying between 1 and 11. Compare the results with your evaluations of the accuracy.
4. Provide a graph that compares the CPU time required by the straightforward and the "Fast" method for N varying between 10^2 and 10^3 for the straightforward and N varying between 10^2 and 10^4 for the "Fast" method. Take $M = 2N$ and the theoretical value of the truncation number that ensures that the required accuracy is achieved.
5. Provide a graph of the abs. max. error (between the standard and fast methods) for N varying between 10^2 and 10^3, $M = 2N$ and the truncation numbers used for each N.

Hints.

1. Note that each source contributes to the error. So the truncation number p, corresponding to a required accuracy ϵ, depends on N . This relationship is an implicit function of p and

you can either solve for p (write a Matlab function to do that) or determine it by developing a table of values and interpolating.

2. Use Matlab.

3. The maximum absolute error is defined as

$$error = \max_{i=1,\ldots,N} \left| v_i^{straightforward} - v_i^{fast} \right|. \qquad \#$$

Plot the theoretical error bound on the same graph (use hint 1).

4. You may keep the truncation number constant (using the one evaluated for $N \leqslant 10^4$) or vary it with N according to the theoretical estimate for the error. In this case the function calculated in hint 1 will be helpful.