Outline

- Representation of functions in the space of coefficients
- Matrix representation of operators
- Truncation and truncated operators
- Translation operator
- Reexpansion coefficients
- R|R and S|S translation operators
- Examples
- S|R and R|S translation operators
- Properties of translation operators
Why do we need represent functions in different spaces?

- Functions should be efficiently summed up;
- Sums of functions should be compressed;
- Error bounds should be established;
- Functions should be translated and expanded over different bases;
- For computations we need discrete and finite function representations.
- Some functions measured experimentally or approximated by splines, and there is no explicit analytical representation in the whole space.

Linear Spaces

\[ a, b, c \in \mathcal{U} \]

1). \[ a + b \in \mathcal{U}; \]
2). \[ a + b = b + a, a + (b + c) = (a + b) + c; \]
3). \[ \exists 0, \ a + 0 = a, \ a + (-a) = a - a = 0; \]
4). \[ \forall \alpha \in \mathbb{C}, \ \alpha a \in \mathcal{U}; \]
5). \[ \forall \alpha, \beta \in \mathbb{C}, \ (\alpha \beta)a = \alpha(\beta)a, \ 1a = a, \]
\[ \alpha(a + b) = \alpha a + \alpha b, \ (\alpha + \beta)a = \alpha a + \beta a. \]
Linear Operators

Linear Spaces

\( \psi \in \mathbb{F}(\Omega), \quad \psi' \in \mathbb{F}(\Omega'), \quad \Omega, \Omega' \subset \mathbb{R}^d, \)

Operator

\( \psi' = \mathcal{A}[\psi], \)

Linear Operator

\[ \mathcal{A}[\alpha \psi_1 + \beta \psi_2] = \alpha \mathcal{A}[\psi_1] + \beta \mathcal{A}[\psi_2], \quad \alpha, \beta \in \mathbb{C}. \]

An example of linear operator: Differential Operator.

Representation of Functions and Operators

Bases

\( \psi \in \mathbb{F}(\Omega), \quad \psi' \in \mathbb{F}(\Omega'), \quad \Omega, \Omega' \subset \mathbb{R}^d. \)

\[ \psi = \sum_n c_n F_n, \quad \psi' = \sum_{n'} c'_{n'} F'_{n'}, \]

\[ \mathcal{A}[F_n] = \sum_{n'} (F(F')_{n'n}) F'_{n'}, \quad \text{Reexpansion Coefficients} \]

\[ \mathcal{A}[\psi] = \mathcal{A} \left[ \sum_n c_n F_n \right] = \sum_n c_n \mathcal{A}[F_n] = \]

\[ = \sum_n c_n \sum_{n'} (F(F')_{n'n}) F'_{n'} = \sum_n \left[ \sum_{n'} (F(F')_{n'n}) c_n \right] F'_{n'} = \sum_n c'_{n'} F'_{n'} = \psi' \]

\[ c'_{n'} = \sum_n (F(F')_{n'n}) c_n, \quad \text{Matrix Representation of operator } \mathcal{A} \]
Function Representation in the Space of Coefficients

Let $\mathbb{F}(\Omega) \subset C(\Omega), \Omega \subset \mathbb{R}^d$, be a normed space of continuous functions with norm

$$\|\Phi(y)\| = \max_{y \in \Omega} |\Phi(y)|.$$  

Let also $\{F_n(y)\}$ be a complete basis in $\mathbb{F}(\Omega)$, so

$$\Phi(y) = \sum_{n=0}^{\infty} a_n F_n(y), \quad y \in \Omega \subset \mathbb{R}^d, \quad \Phi(y), F_n(y) \in \mathbb{F}(\Omega),$$

absolutely and uniformly converges in $\Omega \subset \mathbb{R}^d$. This means that

$$\forall \varepsilon > 0, \quad \exists \rho(\varepsilon), \quad |\Phi(y) - \Phi^\rho(y)| < \varepsilon, \quad \forall y \in \Omega,$$

$$\forall \varepsilon > 0, \quad \exists \rho(\varepsilon), \quad \sum_{n=0}^{\infty} |a_n F_n(y)| < \varepsilon, \quad \forall y \in \Omega,$$

$$\Phi^\rho(y) = \sum_{n=0}^{\rho-1} a_n F_n(y).$$

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Function Representation in the Space of Coefficients (2)

Expansion coefficients can be stacked in an infinite vector

$$A = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_\rho \\ \vdots \end{pmatrix}.$$

Let us denote $A(\Omega)$ a subset of $\mathbb{R}^\rho$ which is an image of $\mathbb{F}(\Omega)$. For any $A \in A(\Omega)$ we request that there exists one-to-one mapping

$$\Phi(y) \equiv A, \quad \Phi(y) \in \mathbb{F}(\Omega), \quad A \in A(\Omega) \subset \mathbb{R}^\rho.$$
p-Truncated Vectors

\[ \forall A \in \mathbb{R}^p, \quad \exists \Phi^p(y) = \sum_{n=0}^{p-1} A_n F_n(y) \in F^p(\Omega) \subset F(\Omega). \]

\( F^p(\Omega) \) is dense in \( F(\Omega) \):

\[ \forall \Phi(y) \in F(\Omega), \quad \exists \Phi^p(y) \in F^p(\Omega), \quad \| \Phi(y) - \Phi^p(y) \| = \sup_{y \in \Omega} |\Phi(y) - \Phi^p(y)| < \varepsilon. \]

Dense in \( F(\Omega) \)

Matrix Representation of Linear Operators

Let \( \Omega' \subset \Omega \) and \( \mathcal{F} \) is a mapping of \( F(\Omega) \) to \( F(\Omega') \). Such mapping can be considered as action of operator \( \mathcal{F} \) on \( \Phi \):

\[ \mathcal{F}[\Phi(y)] = \tilde{\Phi}(y), \quad \Phi(y) \in F(\Omega), \quad \tilde{\Phi}(y') \in F(\Omega') \subset F(\Omega) \]

Respectively, operator \( \mathcal{F} \) generates operator \( F \) that maps the space of expansion coefficients \( \mathbb{A}(\Omega) \rightarrow \mathbb{A}(\Omega') \), which can be considered as representation of the operator \( \mathcal{F} \) in the space of expansion coefficients:

\[ FA = \tilde{A}, \quad A \in \mathbb{A}(\Omega), \quad \tilde{A} \in \mathbb{A}(\Omega') \subset \mathbb{A}(\Omega). \]

Inversely, if we introduce any transform of expansion coefficients \( FA = \tilde{A} \) which provides uniform convergence of function \( \tilde{\Phi}(y') \) corresponding to these coefficients in \( \Omega' \subset \Omega \) then such transform can be treated as operator \( F \) that convert one function from \( F(\Omega) \) to another.

Representation of a Linear Operator
p-Truncation (Projection) Operator

\[ \Pr(p)A = \tilde{A}, \quad A = A(\Omega), \quad \tilde{A} \in A^p(\Omega). \]

\[
A = \begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_{p-1} \\
A_p \\
\vdots
\end{pmatrix} \quad \rightarrow \quad \tilde{A} = \begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_{p-1} \\
0 \\
\vdots
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots
\end{pmatrix}A
\]

In space \( F(\Omega) \):

\[ \Pr(p)[\Phi(y)] = \Phi^p(y), \quad \Phi(y) \in F(\Omega), \quad \Phi^p(y) \in F^p(\Omega), \]

\[ \lim_{p \to \infty} \| \Phi(y) - \Pr(p)[\Phi(y)] \| = 0. \]

**Norm of p-Truncation Operator**

(important for error bounds)

Norm:

\[ \| \Pr(p) \| = \frac{\sup_{y \in \Omega} \| \Pr(p)[\Phi(y)] \|}{\sup_{y \in \Omega} \| \Phi(y) \|}. \]

Triangle inequality:

\[ \| I \| - \| I - \Pr(p) \| \leq \| \Pr(p) \| \leq \| I \| + \| I - \Pr(p) \| = 1 + \| I - \Pr(p) \|. \]

\[ \forall \varepsilon > 0, \quad \exists p, \quad \| I - \Pr(p) \| < \varepsilon, \]

so

\[ \forall \varepsilon > 0, \quad \exists p, \quad 1 - \varepsilon < \| \Pr(p) \| < 1 + \varepsilon, \]
p-Truncated Operator

Let $H : F(\Omega) \rightarrow F(\Omega)$ be an operator, that is represented by infinite matrix

$$H = \begin{pmatrix}
  h_{00} & h_{01} & \ldots & h_{0,p-1} & h_{0p} & \ldots \\
  h_{10} & h_{11} & \ldots & h_{1,p-1} & h_{1p} & \ldots \\
  & \vdots & \ddots & \vdots & \vdots & \ddots \\
  h_{p-1,0} & h_{p-1,1} & \ldots & h_{p-1,p-1} & h_{p-1,p} & \ldots \\
  h_{p,0} & h_{p,1} & \ldots & h_{p,p-1} & h_{pp} & \ldots \\
  & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix}$$

We call operator $H^{(p)} : F(\Omega) \rightarrow F(\Omega)$, $p$ truncated if it is represented by matrix

$$H^{(p)} = \begin{pmatrix}
  h_{00} & h_{01} & \ldots & h_{0,p-1} & 0 & \ldots \\
  h_{10} & h_{11} & \ldots & h_{1,p-1} & 0 & \ldots \\
  & \vdots & \ddots & \vdots & \vdots & \ddots \\
  h_{p-1,0} & h_{p-1,1} & \ldots & h_{p-1,p-1} & 0 & \ldots \\
  0 & 0 & \ldots & 0 & 0 & \ldots \\
  & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix}$$

Norm of p-Truncated Operator
(important for error bounds)

**Theorem:** Let $H : F(\Omega) \rightarrow F(\Omega)$, such that $0 < \|H\| < \infty$, and $H^{(p)} : F(\Omega) \rightarrow F(\Omega)$ is the $p$-truncated operator $H$. Let also $p(\epsilon)$ be such that $1 - \epsilon < \|Pr(p)\| < 1 + \epsilon$. Then

$$(1 - \epsilon)^2 < \|Pr(p)\|^2 = \frac{\|H^{(p)}\|}{\|H\|} = \|Pr(p)\|^2 < (1 + \epsilon)^2,$$

$$\lim_{p \rightarrow \infty} \frac{\|H^{(p)}\|}{\|H\|} = 1.$$

**Proof.**
A $p$-truncated operator can be represented in the form

$$H^{(p)} = Pr(p)HPr(p)$$

(check!)
So the norm of $H^{(p)}$ is

$$\|H^{(p)}\| = \|Pr(p)\|\|H\|\|Pr(p)\| = \|H\|\|Pr(p)\|^2.$$
Translation Operator

Operator $\mathcal{H}(t) : \mathbb{R}(\Omega) \to \mathbb{R}(\Omega'), \Omega \subset \mathbb{R}^d$, $\Omega' \subset \mathbb{R}^d$ is called translation operator corresponding to translation vector $t$, if

$$\mathcal{H}(t)[\Phi(y)] = \Phi(y + t), \quad (y \in \Omega, \quad y + t \in \Omega').$$

Example of Translation Operator
R|R-reexpansion

Let \( x_+ \in \Omega_r(x_+) \subset \mathbb{R}^d \) and \( \Omega_r(x_+) = \{ y \mid x_+ \in r \} \). Let \( x_+ + t \in \Omega_r(x_+) \) and

\[
R_r(y - x_+, t) = \sum_{l=0}^{\infty} \langle R|\mathcal{R} \rangle_{3l}(t) R_l(y - x_+).
\]

Coefficients \( \langle R|\mathcal{R} \rangle_{3l}(t) \) are called \( R|R - reexpansion coefficients \) (regular-to-regular), and infinite matrix

\[
\langle R|\mathcal{R} \rangle(t) = 
\begin{pmatrix}
\langle R|\mathcal{R} \rangle_{00} & \langle R|\mathcal{R} \rangle_{01} & \cdots \\
\langle R|\mathcal{R} \rangle_{10} & \langle R|\mathcal{R} \rangle_{11} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

is called \( R|R - reexpansion matrix \).

Example of R|R-reexpansion

\[
R_m(x) = x^m,
\]

\[
R_m(x + t) = (x + t)^m = x^m + \binom{m}{1}x^{m-1}t + \ldots + \binom{m}{m-1}x^1t^{m-1} + t^m
\]

\[
= \sum_{l=0}^{m} \binom{m}{l} t^l x^{m-l} = \sum_{l=0}^{m} \binom{m}{l} t^l x^{m-l} = \sum_{l=0}^{m} \binom{m}{l} t^l R_l(x).
\]

\[
\langle R|\mathcal{R} \rangle_{3l}(t) = \begin{cases} 
\binom{m}{l} t^l, & l \leq m,
0, & l > m.
\end{cases}
\]
**R|R-transformation operator**

Translation operator $\mathcal{T}(t)$ which is represented in regular basis $\langle R, \langle y - x, \rangle \rangle$ by the $\mathcal{R}|\mathcal{R} - reexpansion matrix$ is called $\mathcal{R}|\mathcal{R}$-transformation operator.

\[
\mathcal{T}(t)[\Phi(y)] = \Phi(y + t)
\]

\[
(\mathcal{R}|\mathcal{R})(t) = \mathcal{T}(t).
\]

---

**Why the same operator named differently?**

\[
\mathcal{T}(t)[\Phi(y)] = \Phi(y + t)
\]

The first letter shows the basis for $\Phi(y)$

\[
\mathcal{T}(t) = \begin{cases}
(\mathcal{R}|\mathcal{R})(t) \\
(\mathcal{S}|\mathcal{S})(t) \\
(\mathcal{C}|\mathcal{C})(t) \\
(\mathcal{E}|\mathcal{E})(t)
\end{cases}
\]

The second letter shows the basis for $\Phi(y + t)$

Needed only to show the expansion basis (for operator representation)
Matrix representation of $R|R$-translation operator

Consider

$$
\Phi(y) = \sum_{n=0}^{\infty} A_n(x_\star) R_n(y-x_\star).
$$

$$
\Phi(y+t) = (R|R)(t)[\Phi(y)] = \sum_{n=0}^{\infty} A_n(x_\star)(R|R)(t)[R_n(y-x_\star)].
$$

$$
= \sum_{n=0}^{\infty} A_n(x_\star) R_n(y-x_\star + t)
$$

$$
= \sum_{n=0}^{\infty} A_n(x_\star) \sum_{j=0}^{\infty} (R|R)_j(t) R_j(y-x_\star)
$$

$$
= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} (R|R)_j(t) A_n(x_\star) \left[R_j(y-x_\star)\right]
$$

$$
= \sum_{j=0}^{\infty} \tilde{A}_j(x_\star, t) R_j(y-x_\star),
$$

$$
\tilde{A}_j(x_\star, t) = \sum_{n=0}^{\infty} (R|R)_j(t) A_n(x_\star), \quad \tilde{A}(x_\star, t) = (R|R)(t)A(x_\star).
$$

Reexpansion of the same function over shifted basis

Compact notation:

$$
\Phi(y) = \sum_{n=0}^{\infty} A_n(x_\star) R_n(y-x_\star) = A(x_\star) \circ R(y-x_\star)
$$

$$
\Phi(y+t) = \sum_{j=0}^{\infty} \tilde{A}_j(x_\star, t) R_j(y-x_\star) = \tilde{A}(x_\star, t) \circ R(y-x_\star)
$$

We have:

$$
\Phi(y) = \Phi((y-t)+t) = \tilde{A}(x_\star, t) \circ R((y-t)-x_\star)
$$

$$
= \tilde{A}(x_\star, t) \circ R(y-x_\star - t).
$$

Also

$$
\Phi(y) = A(x_\star) \circ R(y-x_\star) = A(x_\star + t) \circ R(y-x_\star - t),
$$

so

$$
A(x_\star + t) = \tilde{A}(x_\star, t) = (R|R)(t)A(x_\star).
$$
R|R-reexpansion of the same function over shifted basis (2)

Original expansion
Is valid only here!

Since \( \Omega_{y,t}(x_0+t) \subset \Omega_r(t) \)

Example of power series reexpansion

\[ R_{m}(x) = x^{m}. \]

\[ f(y, x_0) = \sum_{n=0}^{\infty} A_n(x_1, x_0) R_n(y, x_0) = \sum_{n=0}^{\infty} A_n(x_2, x_0) R_n(y, x_2), \]

\[ A(x_1, x_2, x_0) = (R|R)(x_1 - x_0) \cdot A(x_1, x_2). \]

\[
\begin{pmatrix}
A_0(x_1, x_2) \\
A_1(x_1, x_2) \\
A_2(x_1, x_2) \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 2 & (x_1 - x_0)^2 & \cdots \\
0 & 0 & 1 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
A_0(x_1, x_2) \\
A_1(x_1, x_2) \\
A_2(x_1, x_2) \\
\vdots
\end{pmatrix}
\]
S|S-reexpansion

Let $y - x_\ast \in \Omega_\ast(x_\ast) \subset \mathbb{R}^d$, $\Omega_\ast(x_\ast) : |y - x_\ast| > r$, and $\{S_{\ast}(y - x_\ast)\}$ be a singular basis in $C(\Omega)$. Let $y - x_\ast + t \in \Omega_\ast(x_\ast)$ and

$$S_{\ast}(y - x_\ast + t) = \sum_{\alpha = 0}^{\infty} (S|S)_{\alpha}(t) S_{\ast}(y - x_\ast).$$

Coefficients $(S|S)_{\alpha}(t)$ are called $S|S$ - reexpansion coefficients (singular-to-singular), and infinite matrix

$$(S|S)(t) = \begin{pmatrix} (S|S)_{00} & (S|S)_{01} & \cdots \\ (S|S)_{10} & (S|S)_{11} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

is called $S|S$ - reexpansion matrix.

S|S-translation operator

Translation operator $T(t)$ which is represented in singular basis $\{S_{\ast}(y - x_\ast)\}$ by the $S|S$ - reexpansion matrix is called $S|S$-translation operator.

$$T(t)[\Phi(y)] = \Phi(y + t)$$

$$(S|S)(t) = T(t).$$
S|S and R|R-translation operators are very similar,

(actually, this is just two representations of the same translation operator in different domains and bases)

\[
\Phi(y) = B(x_1) \cdot S(y - x_1), \\
\Phi(y + t) = B(x_1 + t) \cdot S(y - x_1) \\
\Phi(y) = B(x_1 + t) \cdot S(y - x_1 - t) \\
\hat{B}(x_1 + t) = (S|S)(t)B(x_1) = B(x_1 + t).
\]

But picture is different…

Original expansion
Is valid only here!

\[|y - x - t| > r = r + |t|\]

Since \(\Omega_{r_1}(x + t) \subset \Omega_r(t)\)!

Also \[|x - x_*| < r\]

singular point!
S|R-reexpansion

Let $y - x_+ \in \Omega_r(x_+) \subset \mathbb{R}^d$, $\Omega_r(x_+)$ : $|y - x_+| < r$, and $\{R_r(y - x_+)\}$ be a regular basis in $C(\Omega_r(x_+))$. Let also $\Omega_s(x_+ - t) : |y - x_+ + t| > R > r$, and $\{S_s(y - x_+ + t)\}$ be a singular basis in $C(\Omega_s(x_+))$, then

$$S_s(y - x_+ + t) = \sum_{l=0}^{\infty} (S|R)_{il}(t)R_l(y - x_+).$$

Coefficients $(S|R)_{il}(t)$ are called $S|R$ - reexpansion coefficients (singular-to-regular), and infinite matrix

$$\begin{pmatrix}
(S|R)_{00} & (S|R)_{01} & \cdots \\
(S|R)_{10} & (S|R)_{11} & \cdots \\
\cdots & \cdots & \cdots \\
\end{pmatrix}$$

is called $S|R$ - reexpansion matrix.

Does R|S reexpansion exist?

- Theoretically yes (in some cases, e.g. analytical continuation);
- In practice, since the domain of S-expansion is larger
then the domain of R-expansion, this either
not useful (due to error bounds), or can be avoided in algorithms;
- We will not use R|S-reexpansions in the FMM algorithms.
S|R-translation operator

Translation operator \( T(t) \) which is represented in singular basis by the \( S|R \) reexpansion matrix is called \( S|R \)-translation operator if the basis of expansion is changed with the translation operation from singular \( S, (y - x_*) \) to regular \( R, (y - x_*, t) \):

\[
T(t)
\begin{bmatrix}
\Phi(y) \\
\end{bmatrix}
= \Phi(y + t);
\]

\[
\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{R}}, \tilde{\mathbf{R}}(t) = T(t).
\]

S|R-operator has almost the same properties as S|S and R|R

(\( t \) cannot be zero)

\[
\Phi(y) = B(x_*) \circ S(y - x_*),
\]

\[
\Phi(y + t) = \tilde{\Lambda}(x_*, t) \circ R(y - x_*)
\]

\[
\Phi(y) = \tilde{\Lambda}(x_*, t) \circ R(y - x_* - t).
\]

\[
\tilde{\Lambda}(x_*, t) = (S|R)(t)B(x_*).
\]
Properties of the translation operator

\[ \mathcal{T}(t)[\Phi(y)] = \Phi(y + t) \]

- **\( \mathcal{T}(0) = \mathcal{T} \) (identity operator).** Proof:
  \[ \mathcal{T}(0)[\Phi(y)] = \Phi(y). \]

- **\( \mathcal{T}(t_1 + t_2) = \mathcal{T}(t_1) \circ \mathcal{T}(t_2) = \mathcal{T}(t_2) \circ \mathcal{T}(t_1) \).** Proof:
  \[ \mathcal{T}(t_1) \circ \mathcal{T}(t_2)[\Phi(y)] = \Phi(y + t_2 + t_1) = \mathcal{T}(t_2 + t_1)[\Phi(y)] = \mathcal{T}(t_1 + t_2)[\Phi(y)]. \]

- **(corollary 1):** \( \mathcal{T}^{-1}(t) = \mathcal{T}(-t) \). Proof:
  \[ \mathcal{T}^{-1}(t) = \mathcal{T}(0) = \mathcal{T}(t - t) = \mathcal{T}(t) \circ \mathcal{T}(-t). \]

- **(corollary 2):** \( \mathcal{T}^n(t) = \mathcal{T}(nt) \). Proof (use induction):
  \[ \mathcal{T}(nt) = \mathcal{T}(n-1)t) \circ \mathcal{T}(t) = \mathcal{T}^{n-1}(t) \circ \mathcal{T}(t) = \mathcal{T}^n(t). \]
Spectrum of the translation operator

\[ T(t)[\Psi(y)] = \lambda \Psi(y), \quad y \in \mathbb{R}^d. \]

Any function of type

\[ \forall \alpha \in \mathbb{R}^d, \quad \Psi(y) = e^{\alpha y}, \quad \lambda = e^{\alpha t}. \]

Check:

\[ T(t)[\Psi(y)] = \Psi(y + t) = e^{\alpha(y + t)} = e^{\alpha t}\rho \Psi(y). \]

Relation to differential operator:

\[ \frac{d\Phi(y)}{ds} = \lim_{|t| \to 0} \frac{\Phi(y + t) - \Phi(y)}{|t|} = \lim_{|t| \to 0} \frac{T(t)[\Phi(y)] - \Phi(y)}{|t|} = \lim_{|t| \to 0} \frac{T(t) - T}{|t|} [\Phi(y)], \quad s = \frac{t}{|t|}. \]

Example from previous lectures

\[ \Phi(y, x_i) = y \frac{1}{x_i}. \]

\[ |y - x_s| < |x_i - x_s| : \quad \text{R-expansion} \]

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} a_m(x_i, x_s) R_m(y - x_s), \]

\[ a_m(x_i, x_s) = -(x_i - x_s)^m, \quad m = 0, 1, \ldots \]

\[ R_m(y - x_s) = (y - x_s)^m, \quad m = 0, 1, \ldots \]

\[ |y - x_s| > |x_i - x_s| : \quad \text{S-expansion} \]

\[ \Phi(y, x_i) = \sum_{m=0}^{\infty} b_m(x_i, x_s) S_m(y - x_s), \]

\[ b_m(x_i, x_s) = (x_i - x_s)^m, \quad m = 0, 1, \ldots \]

\[ S_m(y - x_s) = (y - x_s)^m, \quad m = 0, 1, \ldots \]
In this case we have

\[
|y - x_*| < |t|
\]

\[
S_y(y + x_* t) = (t + y)^{\alpha} \left( t^{m_1} \frac{d^{m_1} S_y(t)}{dt^{m_1}} (y x_*)^{m_2} \right)
- \sum_{m_1}^\infty \frac{1}{m_1!} \frac{d^{m_1} S_y(t)}{dt^{m_1}} R_{m_1}(y x_*) - \sum_{m_1}^\infty (S[R]_{m_1}(t)) R_{m_1}(y x_*).
\]

So

\[
(S[R]_{m_1}(t)) = \frac{1}{m_1!} \frac{d^{m_1} S_y(t)}{dt^{m_1}} = \left( \begin{array}{ccc}
1 & t & t^2 \\
1 & 2t & 3t^2 \\
1 & 3t & 6t^3 \\
& & \ddots
\end{array} \right) \cdot
\]

Norm of the Translation Operator

**Theorem.** Let \( F(\Omega) \) be a set of functions bounded in \( \mathbb{R}^d \). Then \( \| T(t) \| = 1 \).

**Proof.**

\[
\| T(t) \| = \frac{\| T(t) \cdot \Phi(y) \|}{\| \Phi(y) \|} = \frac{\| \Phi(y + t) \|}{\| \Phi(y) \|} = \sup_{y \in \mathbb{R}^d} \frac{|\Phi(y + t)|}{|\Phi(y)|} = 1.
\]
Active and Passive points of view on translation operator

\[ \Phi(y+t) \quad t \rightarrow \quad T(t) \quad \Phi(y) \]

"Active" point of view: Operator transforms function. The reference frame does not change.

"Passive" point of view: Function does not change. Operator transforms the reference frame.

Norms of \( R|\mathcal{R} \), \( S|\mathcal{S} \), and \( S|\mathcal{R} \)-operators (1)

\[ y \quad x_i \quad t \quad y \]

\( \Phi(y) \) is bounded in \( \Omega \).

\( \Omega' \subset \Omega \).

Therefore \( \Phi(y) \) is bounded in \( \Omega' \), and

\[ \| \Phi(y) \|_{\Omega'} = \sup_{y \in \Omega'} |\Phi(y)| \leq \sup_{y \in \Omega} |\Phi(y)| = \| \Phi(y) \|_{\Omega}. \]
Norms of $R|R$, $S|S$, and $S|R$-operators (2)

From the passive point of view, the translation operator does nothing, but just changes the reference frame. So if we consider that $R|R$, $S|S$, and $S|R$ do just change of the reference frame PLUS they shrink the domain, where the function is bounded, then their norms do not exceed 1.

$$\Omega' \subset \Omega$$

$$\| (R|R)(t) \| = \frac{\sup_{y \in \Omega|y}(\Phi(y))}{\sup_{y \in \Omega|y}(\Phi(y))} \leq 1,$$

$$\| (S|S)(t) \| = \frac{\sup_{y \in \Omega|y}(\Phi(y))}{\sup_{y \in \Omega|y}(\Phi(y))} \leq 1,$$

$$\| (S|R)(t) \| = \frac{\sup_{y \in \Omega|y}(\Phi(y))}{\sup_{y \in \Omega|y}(\Phi(y))} \leq 1.$$

Error of exact $R|R$, $S|S$, and $S|R$-translation

If

$$\| \Phi(y) - \Phi^p(y) \| < \epsilon,$$

then

$$\| (R|R)(t)(\Phi(y) - \Phi^p(y)) \| = \| (R|R)(t) \| \| \Phi(y) - \Phi^p(y) \| < \epsilon,$$

$$\| (S|S)(t)(\Phi(y) - \Phi^p(y)) \| = \| (S|S)(t) \| \| \Phi(y) - \Phi^p(y) \| < \epsilon,$$

$$\| (S|R)(t)(\Phi(y) - \Phi^p(y)) \| = \| (S|R)(t) \| \| \Phi(y) - \Phi^p(y) \| < \epsilon.$$