FMM CMSC 878R/AMSC 698R

Lecture 5

Outline

- Far Field Expansions (or S-expansions)
  - Regular Potential (Convergent Series);
  - Regular Potential (Asymptotic Series);
  - Singular Potential;
- Asymptotic Series
- Approaches for Selection of the Basis Functions

Far Field Expansions (S-expansions)

Let $\mathbf{x}_* \in \mathbb{R}^d$.

Might be Singular (at $\mathbf{y} = \mathbf{x}_*$)

We call expansion

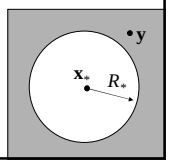
$$\Phi(\mathbf{y}, \mathbf{x}_i) = \sum_{m=0}^{\infty} b_m(\mathbf{x}_i, \mathbf{x}_*) S_m(\mathbf{y} - \mathbf{x}_*)$$

Basis Functions

far field expansion (or S-expansion) outside a sphere

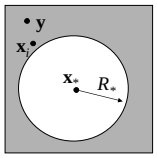
$$|\mathbf{y} - \mathbf{x}_*| > R_*,$$

if the series converges for $\forall \mathbf{y}$, $|\mathbf{y} - \mathbf{x}_*| > R_*$.



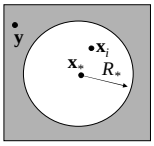
Far Field Expansion of a Regular Potential

…sometimes like this:



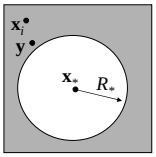
$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$

Can be like this:



$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$

…sometimes like this:



$|\mathbf{x}_i - \mathbf{x}_*| > |\mathbf{y} - \mathbf{x}_*| > R_*$

Local Expansion of a Regular Potential Can be Far Field Expansion Also (Repeat Example from Lecture 3)

Valid for any $r_* < \infty$, and x_i.

$$\Phi(y, x_i) = e^{-(y-x_i)^2} = \sum_{m=0}^{\infty} a_m(x_i, x_*) S_m(y - x_*).$$

We have

$$e^{-(y-x_i)^2} = e^{-[y-x_*-(x_i-x_*)]^2} = e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} e^{2(x_i-x_*)(y-x_*)}$$

$$= e^{-(y-x_*)^2} e^{-(x_i-x_*)^2} \sum_{m=0}^{\infty} \frac{2^m (x_i - x_*)^m (y - x_*)^m}{m!}.$$

Choose

$$a_m(x_i, x_*) = e^{-(x_i-x_*)^2} (x_i - x_*)^m, \quad m = 0, 1, \ldots,$$

$$S_m(y - x_*) = e^{-(y-x_*)^2} \frac{2^m}{m!} (y - x_*)^m, \quad m = 0, 1, \ldots$$

Asymptotic Series

$$f(x, \epsilon) = f_0(x)\varphi_0(\epsilon) + f_1(x)\varphi_1(\epsilon) + f_2(x)\varphi_2(\epsilon) + \ldots = \sum_{n=0}^{\infty} f_n(x)\varphi_n(\epsilon)$$

$$\lim_{\epsilon \to 0} \frac{\varphi_n(\epsilon)}{\varphi_{n+1}(\epsilon)} = 0.$$

Gauge functions

$$f(x, \epsilon) - \sum_{n=0}^{p-1} f_n(x)\varphi_n(\epsilon) = O(f_p(x)\varphi_p(\epsilon))$$

The asymptotic expansion is *uniform* in domain $x \in \Omega$ if

$$\forall x \in \Omega, \quad \left| f(x, \epsilon) - \sum_{n=0}^{p-1} f_n(x)\varphi_n(\epsilon) \right| = O(\varphi_p(\epsilon)).$$

Otherwise the asymptotic expansion is not uniform.

Examples of Uniform and Non-Uniform Expansions

Example of uniform expansion:

$$f(x,\epsilon) = \frac{1}{x+\epsilon}, \quad x > 10$$

$$f(x,\epsilon) = \frac{1}{x}\left(1+\frac{\epsilon}{x}\right)^{-1} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n\epsilon^n}{x^n}$$

Example of non-uniform expansion:

$$f(x,\epsilon) = e^{\epsilon x}, \quad x \in \mathbb{R}^1$$

$$e^{\epsilon x} = \sum_{n=0}^{\infty}\frac{\epsilon^n x^n}{n!}.$$

Prove that! (Hint: consider $x \gg \epsilon^{-1}$).

Example of Far Field Expansion of a Regular Function (Using Asymptotic Series)

$$\Phi(y,x_i) = \frac{1}{1+(y-x_i)^2} = \frac{1}{1+[y-x_*-(x_i-x_*)]^2} = \frac{1}{(y-x_*)^2}\frac{(y-x_*)^2}{1+[y-x_*-(x_i-x_*)]^2}.$$

Let

$$\epsilon = \frac{1}{y-x_*}$$

$$\Phi(\epsilon, x_i-x_*) = \epsilon^2\frac{\frac{1}{\epsilon^2}}{1+\left[\frac{1}{\epsilon}-(x_i-x_*)\right]^2} = \epsilon^2\frac{1}{\epsilon^2+(1-\epsilon x)^2} = \epsilon^2 f(x,\epsilon), \quad x = x_i - x_*$$

$$f(x,\epsilon) = \frac{1}{\epsilon^2+(1-\epsilon x)^2} = \sum_{n=0}^{\infty} f_n(x)\epsilon^n$$

$$f_n(x) = \frac{1}{n!}\frac{\partial^n f(x,\epsilon)}{\partial\epsilon^n}\bigg|_{\epsilon=0}$$

Example of Far Field Expansion of a Regular Function (continuation)

$$f_0(x) = 1,$$
$$f_1(x) = \frac{\partial f(x,\epsilon)}{\partial\epsilon}\bigg|_{\epsilon=0} = 2x,$$
$$f_2(x) = \frac{1}{2!}\frac{\partial^2 f(x,\epsilon)}{\partial\epsilon^2}\bigg|_{\epsilon=0} = 3x^2-1,$$

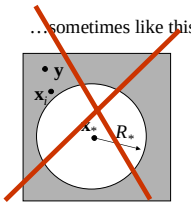
$$\Phi(y,x_i) = \frac{1}{(y-x_*)^2}\sum_{n=0}^{\infty} f_n(x_i-x_*)\frac{1}{(y-x_*)^n}$$

$$y \geqslant 100, \quad x_i = 1, \quad x_* = 0,$$
$$\epsilon = 10^{-2}, \quad x = 1$$

$$\left|\Phi(y,x_i) - \frac{1}{(y-x_*)^2}\left[1+\frac{2(x_i-x_*)}{(y-x_*)}\right]\right| \leqslant \epsilon^4(3x^2-1) = 2\cdot 10^{-8}.$$

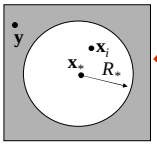
Far Field Expansion of a Singular Potential

…sometimes like this:



$|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$

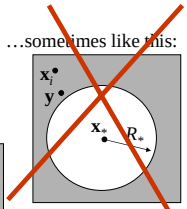
Can be like this:



$|\mathbf{y} - \mathbf{x}_*| > R_* \geq |\mathbf{x}_i - \mathbf{x}_*|$

This case only!

…sometimes like this:



$|\mathbf{x}_i - \mathbf{x}_*| > |\mathbf{y} - \mathbf{x}_*| > R_*$

Example For S-expansion of Singular Potential

$$\Phi(y,x_i) = \frac{1}{y-x_i}.$$

$$\frac{1}{y-x_i} = \frac{1}{y-x_*-(x_i-x_*)} = \frac{1}{(y-x_*)\left[1-\frac{x_i-x_*}{y-x_*}\right]} = \frac{1}{(y-x_*)}\left[1-\frac{x_i-x_*}{y-x_*}\right]^{-1}.$$

$$\left[1-\frac{x_i-x_*}{y-x_*}\right]^{-1} = \sum_{m=0}^{\infty}\frac{(x_i-x_*)^m}{(y-x_*)^m}, \quad |y-x_*| > |x_i-x_*|.$$

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} b_m(x_i,x_*)S_m(y-x_*),$$
$$b_m(x_i,x_*) = (x_i-x_*)^m, \quad m = 0,1,\ldots,$$
$$S_m(y-x_*) = (y-x_*)^{-m-1}, \quad m = 0,1,\ldots$$

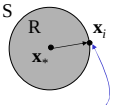
Let us compare with the R-expansion of the same function

$|y-x_*| < |x_i-x_*|$: R-expansion

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} a_m(x_i,x_*)R_m(y-x_*),$$
$$a_m(x_i,x_*) = -(x_i-x_*)^{-m-1}, \quad m = 0,1,\ldots,$$
$$R_m(y-x_*) = (y-x_*)^m, \quad m = 0,1,\ldots$$

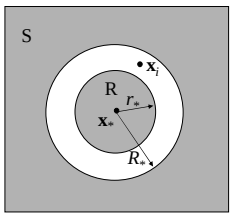
$|y-x_*| > |x_i-x_*|$: S-expansion

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} b_m(x_i,x_*)S_m(y-x_*),$$
$$b_m(x_i,x_*) = (x_i-x_*)^m, \quad m = 0,1,\ldots,$$
$$S_m(y-x_*) = (y-x_*)^{-m-1}, \quad m = 0,1,\ldots$$



Singular Point is located at the Boundary of regions for the R- and S-expansions!

What Do We Need For Real FMM (that provides spatial grouping)



$r_* < R_*$

We need S-expansion for $|\mathbf{y} - \mathbf{x}_*| > R_* > |\mathbf{x}_i - \mathbf{x}_*|$
We need R-expansion for $|\mathbf{y} - \mathbf{x}_*| < r_* < |\mathbf{x}_i - \mathbf{x}_*|$

Basis Functions

- Power series are great, but do they provide the best approximation? (sometimes yes!)
- Other approaches to factorization:
  - Asymptotic Series (Can be divergent!);
  - Orthogonal Bases in L_2;
  - Eigen Functions of Differential Operators;
  - Functions Generated by Differentiation or Other Linear Operators.
- Some of this approaches will be considered in this course.