Wave Field Synthesis

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Abstract

Some background material for the technique called wave-field synthesis is provided. THen we move on to discuss some papers from the literature.

1 Introduction

One way spatially correct reproduction of audio can be achieved is by the spherical array technique we already spoke about in class, and described in (Duraiswami et al, AES, 2005). There the sound field was measured using the array and the coefficients of the spherical wave function expansion, or plane-wave expansion were obtained, and an error bound that ensured the sound field was captured to a certain frequency was obtained. The sound was then transformed to account for scattering off the human by reintroducing HRTF cues, and played back over headphones.

Another approach to reproducing the sound field is via wave-field synthesis. This method starts by assuming that the wave (or in the case of a Fourier transformed version, the Helmholtz equation). It however uses an integral equation statement of the wave equation.

2 Gauss' Divergence theorem and Stokes' theorem

Let $\mathbf{u}$ be a differentiable vector field (function) defined on a domain Ω with boundary S. The flux F of the quantity $\mathbf{u}$ through a surface S with normal

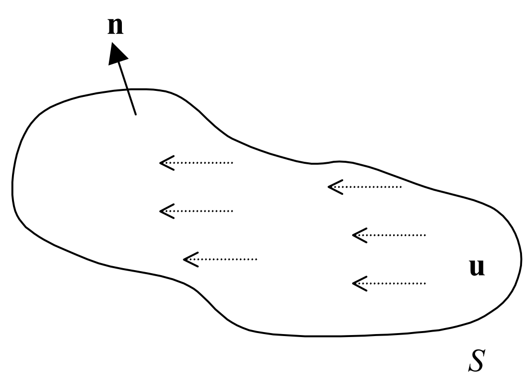
n can be defined as

$$F = \int_S \mathbf{n} \cdot \mathbf{u} dS. \tag{1}$$

Gauss' divergence theorem relates the integral of the divergence of **u** in the domain with its flux through the surface of the domain

$$\int_\Omega \nabla \cdot \mathbf{u} d^3x = \int_S \mathbf{n} \cdot \mathbf{u} dS, \tag{2}$$

where **n** is the normal vector outward to the surface S, that is outward to the domain Ω.



2.1 Green's integral theorems

These theorems play a role analogous to the familiar "integration by parts" in the case of integration over the line, and are fundamental to the derivation of integral equation statements for the Helmholtz and other equations. Recall for integrals over a line, we can write

$$\int_a^b u dv = -\int_a^b v du + (uv)|_a^b \,. \tag{3}$$

Green's first integral theorem states that for a domain Ω with boundary S, given two functions $u(\mathbf{x})$ and $v(\mathbf{x})$ we can write

$$\int_\Omega \left(u\nabla^2 v + \nabla u \cdot \nabla v\right) dV = \int_\Omega \nabla \cdot (u\nabla v)\, dV = \int_S \mathbf{n} \cdot (u\nabla v)\, dS, \tag{4}$$

where we have used the divergence theorem on the quantity $u\nabla v$. This formula may be put into a form that is reminscent of the formula of integration by parts by rearranging terms

$$\int_{\Omega} u\nabla \cdot (\nabla v)\, dV = -\int_{\Omega} \nabla u \cdot \nabla v dV + \int_{S} u\,(\mathbf{n}\cdot\nabla v)\, dS, \tag{5}$$

where we observe that the derivative operator has been exchanged from the function v to the function u, and that the boundary term has appeared.

To derive *Green's second integral theorem*, we write Equation (4) by exchanging the roles of u and v, as

$$\int_{\Omega} \left(v\nabla^2 u + \nabla u \cdot \nabla v\right) dV = \int_{S} v\,(\mathbf{n}\cdot\nabla u)\, dS, \tag{6}$$

and subtract it from Equation (4). This yields

$$\int_{\Omega} \left(u\nabla^2 v - v\nabla^2 u\right) dV \;= \int_{S} \mathbf{n}\cdot(u\nabla v - v\nabla u)\, dS. \tag{7}$$

This equation can also be written as

$$\int_{\Omega} u\nabla^2 v dV = \int_{\Omega} v\nabla^2 u + \int_{S} \mathbf{n}\cdot(u\nabla v - v\nabla u)\, dS. \tag{8}$$

3 Green's function formulation

3.1 Green's function

The free-space Green's function G

$$G(\mathbf{x},\mathbf{y}) = -\frac{\exp(ik\,|\mathbf{x}-\mathbf{y}|)}{4\pi\,|\mathbf{x}-\mathbf{y}|} \tag{9}$$

satisfies the equation

$$\nabla^2 G(\mathbf{x},\mathbf{y}) + k^2 G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \tag{10}$$

for $\mathbf{x}\in\mathbb{R}^3$, and where $\delta(\mathbf{x}-\mathbf{y})$ refers to the Dirac delta function (distribution) which satisfies

$$\int_{\nleq} f(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}) = \left\{ \begin{array}{cl} f(\mathbf{y}) & \text{for } \mathbf{y}\in\Omega \\ 0 & \text{otherwise} \end{array} \right. . \tag{11}$$

This function thus satisfies the Helmholtz equation in the domain $\mathbb{R}^3 \backslash \mathbf{y}$.

This “impulse response" of the Helmholtz equation is a fundamental tool for studying the Helmholtz equation. It is also referred to as the point source solution or the fundamental solution.

3.2 Green’s formula

Let us consider a domain Ω with boundary S. Using the sifting property of the delta function (11) we may write for a given function ϕ at a point $\mathbf{y} \in \Omega$

$$\int_{\not\leq} \phi(\mathbf{x})\, \delta(\mathbf{x}-\mathbf{y})\, d^3\mathbf{x} = \phi(\mathbf{y}), \quad \mathbf{y} \in \Omega. \tag{12}$$

Using Equation (10) the function may be written as

$$\phi(\mathbf{y}) = \int_{\not\leq} \phi(\mathbf{x}) \left[\nabla^2_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) + k^2 G(\mathbf{x},\mathbf{y})\right] d^3\mathbf{x}. \tag{13}$$

Using Green’s second integral theorem we can write the above as

$$\phi(\mathbf{y}) = \int_{\not\leq} k^2 \phi(\mathbf{x})\, G(\mathbf{x},\mathbf{y})\, d^3\mathbf{x} + \int_{\Omega} G(\mathbf{x},\mathbf{y})\, \nabla^2_{\mathbf{x}} \phi d^3\mathbf{x} + \int_{S_{\mathbf{x}}} \mathbf{n} \cdot (\phi(\mathbf{x}) \nabla G - G \nabla \phi(\mathbf{x}))\, dS_{\mathbf{x}} \tag{14}$$

$$= \int_{\Omega} \left[\nabla^2_{\mathbf{x}} \phi(\mathbf{x}) + k^2 \phi(\mathbf{x})\right] G(\mathbf{x},\mathbf{y})\, d^3\mathbf{x} + \int_{S_{\mathbf{x}}} \mathbf{n} \cdot (\phi(\mathbf{x}) \nabla G - G \nabla \phi(\mathbf{x}))\, dS_{\mathbf{x}}. \tag{15}$$

Let us assume that the the function $\phi(\mathbf{x})$ satisfies

$$\nabla^2 \phi + k^2 \phi = f. \tag{16}$$

Then we see that the solution to this equation can be written as

$$\phi(\mathbf{y}) = \int_{\Omega} f(\mathbf{x})\, G(\mathbf{x},\mathbf{y})\, d^3\mathbf{x} + \int_{S_{\mathbf{x}}} \mathbf{n} \cdot (\phi(\mathbf{x}) \nabla G - G \nabla \phi(\mathbf{x}))\, dS_{\mathbf{x}}. \tag{17}$$

If the domain has no boundaries, we see that the solution to the problem is obtained as a convolution of the right hand-side with the impulse response, i.e., the Green’s funtion

$$\phi(\mathbf{y}) = \int_{\Omega} f(\mathbf{x})\, G(\mathbf{x},\mathbf{y})\, d^3\mathbf{x}. \tag{18}$$

Let us consider the case when ϕ satisfies the Helmholtz equation, i.e., Equation (16) with $f = 0$. Then this formula provides us with the solution for ϕ in the domain from its boundary values

$$\phi(\mathbf{y}) = \int_{S_{\mathbf{x}}} \left(\phi(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_x} - G(\mathbf{x}, \mathbf{y}) \frac{\partial \phi}{\partial n}(\mathbf{x}) \right) dS_{\mathbf{x}}. \tag{19}$$

This equation is valid for the case when $\mathbf{y}$ is in the domain (and not on the boundary). This function satisfies the Sommerfeld condition as $|\mathbf{y}| \to \infty$. This equation is also called the Helmholtz integral equation, or the Kirchhoff integral equation.

$$\phi(\mathbf{y}) = \int_{S_{\mathbf{x}}} \left(\phi(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_x} - G(\mathbf{x}, \mathbf{y}) \frac{\partial \phi}{\partial n_x}(\mathbf{x}) \right) dS_{\mathbf{x}}, \quad \mathbf{y} \text{ inside the domain.}$$

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$$p(\mathbf{y}) = \int_{S_{\mathbf{x}}} \left(a \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_x} - bG(\mathbf{x}, \mathbf{y}) \right) dS_{\mathbf{x}}, \quad \mathbf{y} \text{ inside the domain.}$$

Consider now a room whose walls contain loud speakers. Suppose we wish to create a given field at a point $\mathbf{y}$. Then Kirchoff integral equation says that we can do this by setting the field at the boundary appropriately. IN Wave Field Synthesis one of these items is assumed as zero (the dipoles), while the loudspeaker is set to create monopoles of appropriate strengths.

5 Practical WFS

In practice the whole space boundary cannot be covered with speakers of two types (monopole and dipole), or even one type. Often some speakers are introduced at particular locations outside a region where the sound is to be reproduced. Then, the speaker outputs are modified to reproduce certain measurements in another space. Thus the relevance of of the above derivation is questionable. However, since it is often presented as a justification for the WFS concept, and since this is classical material that is good to know, we have presented it here.

We will consider four papers from the literature which will give a flavor of current research in this area.