The Wave and Helmholtz Equations

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Abstract

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1 Acoustic Waves

1.1 Barotropic Fluids

The usual assumptions for acoustic problems are that acoustic waves are perturbations of the medium (fluid) density $\rho(\mathbf{r},t)$, pressure $p(\mathbf{r},t)$, and mass velocity, $\mathbf{v}(\mathbf{r},t)$, where t is time. It is also assumed that the medium is inviscid, and that perturbations are small, so that

$$\rho = \rho_0 + \rho', \quad p = p_0 + p', \quad \rho' \ll \rho_0, \quad p' \ll p_0, \quad |\mathbf{v}'| \ll c \sim \sqrt{\frac{p_0}{\rho_0}}. \tag{1}$$

Here the perturbations are near an initial spatially uniform state (ρ_0, p_0) of the fluid at rest ($\mathbf{v}_0 = \mathbf{0}$) and are denoted by primes. The latter equation states that the mass velocity of the fluid is much smaller than the speed of sound c in that medium. In this case the linearized continuity (mass conservation) and momentum conservation equations can be written as

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}') = 0, \quad \rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0, \tag{2}$$

where

$$\nabla = \mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z}, \tag{3}$$

is the invariant "nabla" operator, represented by formula (3) in Cartesian coordinates, where $(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z)$ are the Cartesian basis vectors.

Differentiating the former equation with respect to t and excluding from the obtained expression $\partial \mathbf{v}'/\partial t$ due to the latter equation, we obtain

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p'. \tag{4}$$

Note now that system (2) is not closed since the number of variables (three components of velocity, pressure, and density) is larger than the number of equations. The relation needed to close the system is equation of state, which relates perturbations of the pressure and density. The simplest form of this relation is provided by *barotropic* fluids, where the pressure is a function of density only:

$$p = p(\rho). \tag{5}$$

We can expand this in series near the unperturbed state

$$p = p(\rho_0) + \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} (\rho - \rho_0) + O\left((\rho - \rho_0)^2\right). \tag{6}$$

Taking into account that $p(\rho_0) = p_0$, we obtain neglecting the second-order nonlinear term:

$$p' = c^2 \rho', \quad c^2 = \left. \frac{\mathrm{d}p}{\mathrm{d}\rho} \right|_{\rho=\rho_0}, \tag{7}$$

where we used definition of the speed of sound in the unperturbed fluid, which is a real positive constant (property of the fluid). Substitution of expression (7) into relation (4) yields the *wave equation* for pressure perturbations

$$\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = \nabla^2 p'. \tag{8}$$

Obviously, the density perturbations satisfy the same equation. The velocity is a vector and satisfies the *vector* wave equation:

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{v}'}{\partial t^2} = \nabla^2 \mathbf{v}'. \tag{9}$$

This also means that each of the components of the velocity $\mathbf{v}' = \left(v'_x, v'_y, v'_z\right)$ satisfies the *scalar* wave equation (8). Note that these components are not independent. The momentum equation (2) shows that there exists some scalar function ϕ', which is called the *velocity potential*, such that

$$\mathbf{v}' = \nabla \phi, \quad \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi \quad \left(\rho_0 \frac{\partial \phi}{\partial t} = -p' \right). \tag{10}$$

So the problem can be solved for the potential and then the velocity field can be found as the gradient of this scalar field.

1.2 Isentropic Fluids

The analysis that was done in class in Lecture 2, corresponds to the case of an isentropic gas. In this case as was noted in class, the equations that are finally derived are the same, except that the sound velocity is

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}},$$

with γ the ratio of specific heats for the gas.

1.3 Fourier Transform

The wave equation derived above is linear and obey particular solutions periodic in time. Particularly, if the time dependence is a harmonic function of *circular frequency* ω we can write

$$\phi(\mathbf{r},t) = \mathrm{Re}\left(e^{-i\omega t} \psi(\mathbf{r})\right), \quad i^2 = -1, \tag{11}$$

where $\psi(\mathbf{r})$ is some complex valued scalar function and the real part is taken, since $\phi(\mathbf{r},t)$ is real. Substituting expression (11) into the wave equation, (10), we can see that the latter is satisfied, if $\psi(\mathbf{r})$ is a solution of the Helmholtz equation

$$\nabla^2 \psi(\mathbf{r}) + k^2 \psi(\mathbf{r}) = 0, \quad k = \frac{\omega}{c}. \tag{12}$$

The constant k is called the *wavenumber* and is real for real ω. The name is related to the case of plane wave propagating in the fluid, where the *wavelength* is $\lambda = 2\pi/k$, and so k is the *number* of waves per 2π.

The Helmholtz equation stands therefore for monochromatic waves, or waves of some given frequency ω. For *polychromatic* waves, or sums of waves of different frequencies, we can sum up solutions with different ω. More generally, we can perform the *inverse Fourier transform* of potential $\phi(\mathbf{r},t)$ with respect to the temporal variable:

$$\psi(\mathbf{r},\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \phi(\mathbf{r},t)\, \mathrm{d}t. \tag{13}$$

In this case $\psi(\mathbf{r},\omega)$ satisfies the Helmholtz equation (12). Solving this equation we can determine solution of the wave equation using the *forward Fourier transform*:

$$\phi(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\psi(\mathbf{r},\omega)\,\mathrm{d}\omega, \quad \omega = ck. \tag{14}$$

We note that in the Fourier transform the frequency ω can be either negative or positive. This results in either negative or positive values of the wavenumber. However the Helmholtz equation depends on k^2 and is *invariant* with respect to change of the sign of k. This phenomenon, in fact, has a deep physical and mathematical origin, and appears from the property of the wave equation to be a *two wave equation*. It describes solutions which are a superposition of two waves propagating with the same velocity in *opposite directions*. We will consider this property and rules for selection of proper signs later in this chapter, in the section dedicated to boundary conditions.

In the case of Fourier transform we can state that the Helmholtz equation is the wave equation in the *frequency domain*. Since methods for fast Fourier transform are widely available, conversion from time to frequency domain and back are computationally efficient, and so the problem of solution of the wave equation can be reduced to solution of the Helmholtz equation, which is an equation of lower dimensionality (3 instead of 4) than the wave equation.

2 Boundary Conditions

The Helmholtz equation is an equation of the elliptic type, for which it is usual to consider boundary value problems. Boundary conditions follow from particular physical laws (conservation equations) formulated on the boundaries of the domain in which solution is required. This domain can be finite (internal problems) or infinite (external problems). For infinite domains solutions should satisfy some conditions at the infinity. These conditions also have a physical origin. For the Helmholtz equation that arises as a transform of the wave equation into the frequency domain the boundary conditions should be understood in the context of the original wave equation.

2.1 Conditions at Infinity

2.1.1 Spherically Symmetrical Solutions

To understand conditions which should be imposed for solutions of the Helmholtz equation in infinite domains we start with the consideration of spherically symmetrical solutions of scalar wave equation. In this case the dependence on $\mathbf{r}$ of a function ϕ, which satisfies the wave equation (10), is the dependence on the distance $r = |\mathbf{r}|$ only. It is well known that solution of this equation can be written in the following form

$$\phi(r,t) = \frac{1}{r}\left[f(t + r/c) + g(t - r/c)\right], \tag{15}$$

where f and g are two arbitrary differentiable functions. The former function describes *incoming waves* to the center $r = 0$ and the latter function describes *outgoing waves* from the center $r = 0$. Indeed the incoming wave phase can be characterized by some constant value of f, which is realized at $r = -ct+\text{const}$, and so the wavefronts converge to the center as t is growing. Inversely, the outgoing wave phase is characterized by some constant value of g, which is realized at $r = ct+\text{const}$ and so the wavefronts for the outgoing waves diverge from the center as increasing t.

Therefore spherically symmetrical solution of the scalar wave equation can be characterized by specification of two functions of time $f(t)$ and $g(t)$. Assume that these functions satisfy necessary conditions to perform the Fourier transform. Then, in the frequency domain we have images of these functions according (10):

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\mathrm{d}t, \quad \widehat{g}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} g(t)\mathrm{d}t. \tag{16}$$

With these definitions and solution (15) we can determine the image, or phasor $\psi(r,\omega)$, of $\phi(r,t)$ in the frequency domain as

$$\begin{aligned}\psi(r,\omega) &= \int_{-\infty}^{\infty} e^{i\omega t}\phi(r,t)\,\mathrm{d}t = \frac{1}{r}\left[\int_{-\infty}^{\infty} e^{i\omega t} f(t+r/c)\mathrm{d}t + \int_{-\infty}^{\infty} e^{i\omega t} g(t-r/c)\mathrm{d}t\right] \\ &= \frac{1}{r}\left[\int_{-\infty}^{\infty} e^{i\omega\left(t'-r/c\right)} f(t')\mathrm{d}t' + \int_{-\infty}^{\infty} e^{i\omega\left(t'+r/c\right)} g(t')\mathrm{d}t\right] \\ &= \frac{1}{r}\widehat{f}(\omega)\,e^{-ikr} + \frac{1}{r}\widehat{g}(\omega)\,e^{ikr}, \quad k = \frac{\omega}{c}.\end{aligned} \tag{17}$$

Here we defined $k = \omega/c$ and so this quantity is negative for $\omega < 0$ and positive for $\omega > 0$. The function $\psi(r,\omega)$ is a solution of the spherically symmetrical Helmholtz equation (12). It is seen that solutions corresponding to the incoming waves are proportional to e^{-ikr}, while solutions corresponding to the outgoing waves are proportional to e^{ikr}.

It is not difficult to see also that at large r we have

$$r\left(\frac{\partial\psi}{\partial r} - ik\psi\right) = -2ik\widehat{f}(\omega)\,e^{-ikr} + O\left(\frac{1}{r}\right), \tag{18}$$

$$r\left(\frac{\partial\psi}{\partial r} + ik\psi\right) = 2ik\widehat{g}(\omega)\,e^{ikr} + O\left(\frac{1}{r}\right). \tag{19}$$

This means that if the condition

$$\lim_{r\to\infty}\left[r\left(\frac{\partial\psi}{\partial r} - ik\psi\right)\right] = 0 \tag{20}$$

holds then $\widehat{f}(\omega) \equiv 0$. This results in $f(t) \equiv 0$ and in this case $\phi(r,t)$ consists only of outgoing waves. Similarly, in the case if condition

$$\lim_{r\to\infty}\left[r\left(\frac{\partial\psi}{\partial r} + ik\psi\right)\right] = 0 \tag{21}$$

holds then solution consists only of incoming waves and $g(t) \equiv 0$.

2.1.2 Sommerfeld Radiation Condition

The problems, which are usually considered in relation with the wave equation in three dimensional unbounded domains are scattering problems. In this case the wave function is specified as

$$\phi(\mathbf{r},t) = \phi_{in}(\mathbf{r},t) + \phi_{scat}(\mathbf{r},t), \tag{22}$$

where both functions $\phi_{in}(\mathbf{r},t)$ and $\phi_{scat}(\mathbf{r},t)$ satisfy the wave equation. Function $\phi_{in}(\mathbf{r},t)$ is the potential of the incident field, while $\phi_{scat}(\mathbf{r},t)$ is the potential of the scattered field, which arises due to the presence of one or several scatterers. In the absence of scatterers $\phi(\mathbf{r},t) = \phi_{in}(\mathbf{r},t)$ is some given function (e.g. the potential of a plane wave propagating along the z direction, $\phi_{in}(\mathbf{r},t) = F(t - z/c)$).

To understand the scattered field we may turn our attention to the *Huygens principle*, which represents wave propagation as emission of secondary wave from the points located on the current wavefront. When the primary wave described by $\phi_{in}(\mathbf{r},t)$ reaches the scatterer boundary the secondary waves are generated from the boundary points located at the intersection of the boundary and the wavefront. Due to finite speed of wave propagation spatial points far from the boundary "do not know" about these secondary waves, so these waves can be thought as waves *outgoing* from the boundary points. For each point then we can write the secondary wave potential in the form (15), where $f \equiv 0$ and, therefore, in the frequency domain condition (20) holds. Since the total scattered field, $\phi_{scat}(\mathbf{r},t)$, can be seen now as a superposition of outgoing waves, corresponding potential in the frequency domain should satisfy condition

$$\lim_{r\to\infty}\left[r\left(\frac{\partial\psi_{scat}}{\partial r} - ik\psi_{scat}\right)\right] = 0. \tag{23}$$

This condition is called *Sommerfeld radiation condition* or just *radiation condition.* It states that the scattered field consists of outgoing waves only. Solutions of the Helmholtz equation which satisfy the radiation condition are called *radiating solutions* or *radiating functions.*

In some wave problems considered in infinite domains all the wave sources and scatterers can be enclosed inside some sphere. Since in the absence of the wave sources solution of the wave equation is trivial, $\phi(\mathbf{r},t) \equiv 0$, then all perturbations for points located outside the sphere come only from some events inside the sphere. This means that in this case the total field in the frequency domain, $\psi(\mathbf{r},\omega)$, is a radiating function.

We emphasize that the radiation condition (23) derived from consideration of point sources is applied to a set of sources, i.e. to the case $\psi_{scat} = \psi_{scat}(\mathbf{r},k)$. Generally, the far field asymptotics of ψ_{scat} is

$$\psi_{scat} \sim \frac{1}{r}\Psi(\theta,\varphi)\, e^{ikr}, \tag{24}$$

where $\Psi(\theta,\varphi)$ is the angular dependence on spherical angles θ and φ, and so condition (23) holds. Indeed, from a very remote point a set of sources or scatterers is seen as one point (like we see galaxies consisting of many stars as one "star"). While for different angles there will be different values of of ψ_{scat} (so it is not spherically symmetrical), for a given, or fixed, angles θ and φ there is no difference between the asymptotic behavior of a set of sources and an equivalent single source.

2.2 Transmission Conditions

Real systems can be considered as a unity of domains occupied by relatively homogeneous media. While the physical properties of different substances can differ substantially (say air and rigid particles) one should keep in mind that waves of different nature can propagate in any substance (e.g. acoustic waves) and therefore wave-type equations can be used for their description. Due to the difference in the properties the speed of propagation of perturbations is different for different media. Therefore for descriptions of waves in each domain we can apply the wave equation with the speed of sound corresponding to the medium that occupies that domain. The problem then is to provide sufficient conditions on the domain boundaries, that enable to match solutions in different domains and build solutions for the corresponding wave equation. These conditions are known as *transmission* conditions, which also can be interpreted as *jump conditions* or conditions for discontinuities, since the wave function and/or its derivatives jump on the contact boundaries. In general, the jump conditions can be derived from the same conservation equations that lead to the governing equations. The form of these conservation laws should be written in integral form to allow discontinuities and then the conditions arise after shrinking the domain to the contact surfaces.

In acoustics we usually consider problems, when the boundaries of the domains are either immovable or move with velocities much smaller than the speed of sound. We also consider the case when the amplitude of pressure perturbations is small and perturbations of the mass velocity are small as well. In the linear approximation, this results in the following two conditions on a contact surface S with normal $\mathbf{n}$ separating two media marked as 1 and 2 :

$$\mathbf{v}_1' \cdot \mathbf{n} = \mathbf{v}_2' \cdot \mathbf{n}, \quad p_1' = p_2'. \tag{25}$$

The first condition states that the normal velocities to the surface are the same. In fluid mechanics this is known as kinematic condition. In fact, it follows from the mass conservation equation in assumption that there is no mass transfer through the surface S. The second condition, sometimes called dynamic conditions, follows from the momentum conservation equation and is valid if there are no surface forces. As follows from this description these conditions should be modified if mass is transferred through the surface (say due to phase transitions), and if there are some appreciable surface forces (for example, surface tension). These conditions are sufficient to match solutions of the wave or Helmholtz equation in two domains.

Depending on the problem to be solved (wave equation for pressure or for the velocity potential) conditions (25) can be written in terms of pressure or velocity potential and their derivatives only. Consider first the pressure equations. As follows from the momentum conservation equation (2) written in the frequency domain, the phasors of pressure and velocity perturbations, $\widehat{p}'$ and $\widehat{\mathbf{v}}'$, satisfy equations

$$-i\omega\rho_1\widehat{\mathbf{v}}_1' + \nabla\widehat{p}_1' = \mathbf{0}, \quad -i\omega\rho_2\widehat{\mathbf{v}}_2' + \nabla\widehat{p}_2' = \mathbf{0}. \tag{26}$$

Taking the scalar product of these equations with normal $\mathbf{n}$, denoting the normal derivative $\partial/\partial n = \mathbf{n}\cdot\nabla$, and noticing that relations (25) also holds in the frequency domain (remember our assumption that the speed of the boundary is much smaller than the speed of sound!), we obtain the following transmission conditions for pressure perturbations applicable to matching solutions of the Helmholtz equation in domains 1 and 2:

$$\frac{1}{\rho_1}\frac{\partial\widehat{p}_1'}{\partial n} = \frac{1}{\rho_2}\frac{\partial\widehat{p}_2'}{\partial n}, \quad \widehat{p}_1' = \widehat{p}_2'. \tag{27}$$

Here ρ_1 and ρ_2 are the respective medium densities.

Now consider the problem formulation for the Helmholtz equation written in terms of the velocity potential (10). The integral of momentum equation (expression in the parentheses in equation (10)) can be written in the phasor space, where we use notation ψ for the phasor of ϕ (see (13)) as

$$i\omega\rho_1\psi_1 = p_1', \quad i\omega\rho_2\psi_2 = p_2'. \tag{28}$$

Hence, relations (25) lead to the following transmission conditions:

$$\frac{\partial\psi_1}{\partial n} = \frac{\partial\psi_2}{\partial n}, \quad \rho_1\psi_1 = \rho_2\psi_2. \tag{29}$$

Comparing these conditions with relation (27) we can see that in case of pressure formulation the function which satisfies Helmholtz equations in two different domains is continuous, while its normal derivatives have a discontinuity. The opposite situation, when the normal derivative is continuous, while the wave function itself has a jump on the boundary is the case for formulation of the same acoustic problem in terms of the velocity potential.

Note that for acoustic waves in complex media (dispersion, dissipation, relaxation) conditions (27) and (29) should be modified according to the model of the media. Proper transmission conditions in this case can be obtained from general mass and momentum conservation relations (25) and specific equations of state for the medium, such as (**??**), written in the frequency domain.

2.3 Conditions on the Boundaries

Conditions on the boundaries of the domain 1 are used when either the properties of the boundary material (medium 2) are very different from the properties of the medium 1 or can be modelled or assumed. In the former case the transmission conditions can be simplified and provide sufficient conditions for solution of the Helmholtz equation. In the latter case simplification of general problem usually follow from consideration of some model problem by applying the results to more general case. Since such modeling is out of scope of this book, we mention here the following basic types of boundary conditions for scalar wave equation and Maxwell equations. Here we assume that the domain of consideration is medium 1 (we also call it as host, carrier, or just a medium with no index), and the material of the boundary has properties of medium 2 (we will drop the indexing if it is clear from the context). The normal derivative everywhere is taken inward the domain of the carrier medium (direction from medium 2 to medium 1).

- The Dirichlet boundary condition:

$$\psi|_S = 0. \tag{30}$$

  This condition appears, e.g. for complex amplitude pressure in acoustics, when the material of the surface has very low acoustic impedance compared to the acoustic impedance of the carrier medium ($\rho_2 c_2 \ll \rho_1 c_1$). In this case the surface is called *sound soft*.

- The Neumann boundary condition:
$$\left.\frac{\partial\psi}{\partial n}\right|_S = 0. \tag{31}$$
In acoustics this condition holds for complex amplitude of pressure, when the surface material has much higher acoustic impedance than the acoustic impedance of the host medium ($\rho_2 c_2 \gg \rho_1 c_1$). In this case the surface is called *sound hard*.

- The Robin (or mixed, or impedance) boundary condition:
$$\left.\left(\frac{\partial\psi}{\partial n} + i\sigma\psi\right)\right|_S = 0. \tag{32}$$
In acoustics this condition is used to model finite acoustic impedance of the boundary. In this case σ is the admittance of the surface. Solutions of the Helmholtz equation with the Robin boundary condition in limiting cases $\sigma \to 0$ and $\sigma \to \infty$ turns into solutions of the same equation with the Neumann and Dirichlet boundary conditions, respectively.

The boundary value problems with those conditions are called the Dirichlet, Neumann, and the Robin problems, respectively.

3 Solutions of 1-D problems

Let us consider the one dimensional wave equation