

# Separation of Variables

Some slides adapted from Ake  
Nordlund and Dennis Healy

# Separation of variables ...

- Main principles
  - Why?
- Techniques
  - How?

# Main principles

- Why?
  - Because many systems in physics are separable
  - In particular systems with symmetries
  - Such as spherically symmetric problems

Example: Spherical harmonics

$$u(r, \Theta, \varphi, t) = Y_{lm}(r, \Theta, \varphi) e^{i\omega t}$$

# Partial Differential Equations: Separation of Variables ...

- Main principles
  - Why?
    - Common in physics!
    - Greatly simplifies things!
- Techniques
  - How?

# Separation of variables: the general method

- Suppose we seek a PDE solution  $u(x,y,z,t)$

Ansatz:

$$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\partial u / \partial x = X'(x) Y(y) Z(z) T(t)$$

$$\partial^2 u / \partial x^2 = X''(x) Y(y) Z(z) T(t)$$

# Separation of variables: the general method

- Suppose we seek a PDE solution  $u(x,y,z,t)$ :

Ansatz:

$$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

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$$\partial u / \partial y = X(x) Y'(y) Z(z) T(t)$$

$$\partial^2 u / \partial y^2 = X(x) Y''(y) Z(z) T(t)$$

# Separation of variables: the general method

- Suppose we seek a PDE solution  $u(x,y,z,t)$ :

Ansatz:

$$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\partial u / \partial z = X(x) Y(y) Z'(z) T(t)$$

$$\partial^2 u / \partial z^2 = X(x) Y(y) Z''(z) T(t)$$

# Separation of variables: the general method

- Suppose we seek a PDE solution  $u(x,y,z,t)$ :

Ansatz:

$$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\partial u / \partial t = X(x) Y(y) Z(z) T'(t)$$

$$\partial^2 u / \partial t^2 = X(x) Y(y) Z(z) T''(t)$$

# Examples

$$u(x,y,z,t) = xyz^2 \sin(bt)$$

$$u(x,y,z,t) = xy + zt$$

$$u(x,y,z,t) = (x^2 + y^2) z \cos(\omega t)$$

# Examples

Separable?

✓

$$u(x,y,z,t) = xyz^2 \sin(bt)$$

Yes!

~~$$u(x,y,z,t) = xy + zt$$~~

No!

?

$$u(x,y,z,t) = (x^2+y^2)z \cos(\omega t)$$

Hm...

# Examples

Separable?

✓

$$u(x,y,z,t) = xyz^2 \sin(bt)$$

Yes!

~~$$u(x,y,z,t) = xy + zt$$~~

No!

?

$$u(x,y,z,t) = (x^2+y^2) z \cos(\omega t)$$

Hm...

$$(x^2+y^2) \Rightarrow p^2$$

# Examples

Separable?

✓

$$u(x,y,z,t) = xyz^2 \sin(bt)$$

Yes!

~~$$u(x,y,z,t) = xy + zt$$~~

No!

✓

$$u(p,z,t) = p^2 z \cos(\omega t)$$

Yes!

# Example PDEs

- The wave equation

$$c^2 \nabla^2 u = \partial^2 u / \partial t^2$$

where

$$\nabla^2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$$

Ansatz:

$$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

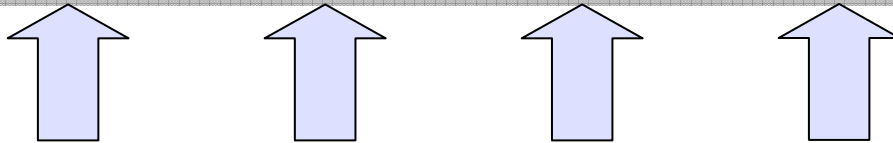
# Separation of variables in the wave equation

- For the ansatz to work we must have (lets set  $c=1$  for now to get rid of a triviality!)

$$X''YZT + XY''ZT + XYZ''T = XYZT''$$

- Divide through with  $XYZT$ , for:

$$X''/X + Y''/Y + Z''/Z = T''/T$$



This can only work if all of these are constants!

# Separation of variables in the wave equation

- Let's take all the constants to be negative
  - Negative  $\Rightarrow$  bounded behavior at large  $x, y, z, t$

$$X''/X + Y''/Y + Z''/Z = T''/T$$

These are called **separation constants**

$$- \bullet^2 \quad - \circ^2 \quad - \blacksquare^2 = - \mu^2$$

This is typical: If / when a PDE allows separation of variables, the partial derivatives are replaced with ordinary derivatives, and all that remains of the PDE is an algebraic equation and a set of ODEs – much easier to solve!

# Separation of variables in the wave equation

- Solutions of the ODEs:

Example: x-direction

$$X''/X = -\bullet^2$$

General solution:

$$X(x) = A \exp(i \bullet x) + B \exp(-i \bullet x)$$

or

$$X(x) = A' \cos(\bullet x) + B' \sin(\bullet x)$$

# Separation of variables in the wave equation

- Combining all directions (and time)

Example:

$$u = \exp(i \bullet x) \exp(i \circ y) \exp(i \blacksquare z) \exp(-i c \mu t)$$

or

$$u = \exp(i (\bullet x + \circ y + \blacksquare z - \omega t))$$

This is a traveling wave, with wave vector  $\{\bullet, \circ, \blacksquare\}$  and frequency  $\omega$ . A general solution of the wave equation is a super-position of such waves.

# Example PDEs

- The Laplace equation

$$\kappa \nabla^2 u = 0$$

where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

Ansatz:

$$u(x,y,z) = X(x) Y(y) Z(z)$$

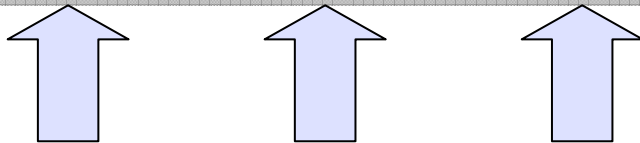
# Separation of variables in the Laplace equation

- For the ansatz to work we must have

$$X''YZ + XY''Z + XYZ'' = 0$$

- Divide through with  $XYZ$ , for:

$$X''/X + Y''/Y + Z''/Z = 0$$

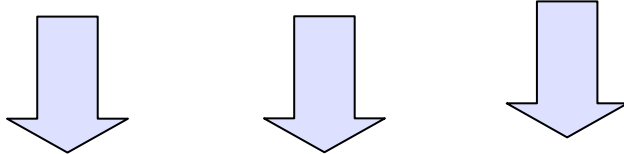


Again, this can only work if all of these are constants!

# Separation of variables in the Laplace equation

- This time we cannot take all constants to be negative!

$$X''/X + Y''/Y = 0$$



$$- \bullet^2 \quad - \circ^2 = 0$$

**At least one** of the terms must be positive  $\Rightarrow$  imaginary wave number; i.e., **exponential** rather than sinusoidal behavior!

# Separation of variables in the Laplace equation

- Solutions of the ODEs:

Example: x-direction

$$X''/X = -\bullet^2 < 0 \quad \text{assumed}$$

General solution:

$$X(x) = A \exp(i \bullet x) + B \exp(-i \bullet x)$$

or

$$X(x) = A' \cos(\bullet x) + B' \sin(\bullet x)$$

# Separation of variables in the Laplace equation

- Solutions of the ODEs:

Example: y-direction

$$Y''/Y = + \bullet^2 > 0 \quad \text{follows!}$$

General solution:

$$Y(y) = C \exp(\bullet y) + D \exp(-\bullet y)$$

or

$$Y(y) = C' \cosh(\bullet y) + B' \sinh(\bullet y)$$

# Separation of variables in the Laplace equation

- Combining x & y directions

Example:

$$u = [A \exp(i \bullet x) + B \exp(-i \bullet x)] [C \exp(\bullet y) + D \exp(-\bullet y)]$$

or

$$u = [A' \cos(\bullet x) + B' \sin(\bullet x)] [C' \cosh(\bullet y) + D' \sinh(-\bullet y)]$$

# Message

- Complex exponentials are a natural basis for representing the solutions of the wave equation
- A microphone samples the solution at a given point
- Variation in time is naturally expressed in terms of Fourier series.

# Fourier Methods

- Fourier analysis (“harmonic analysis”) a key field of math
- Has many applications and has enabled many technologies.
- Basic idea: Use Fourier representation to represent functions
- Has fast algorithms to manipulate them (the fast Fourier Transform)

# Basic idea

- Function spaces can have many different types of bases
- We have already met monomials and other polynomial basis functions
- Fourier introduced another set of basis functions : the Fourier series
- These basis functions are particularly good for describing things that repeat with time

# Fourier's Representation

$$F(t) = A_0/2 + A_1 \cos(t) + A_2 \cos(2t) + A_3 \cos(3t) + \dots \\ + B_1 \sin(t) + B_2 \sin(2t) + B_3 \sin(3t) + \dots$$

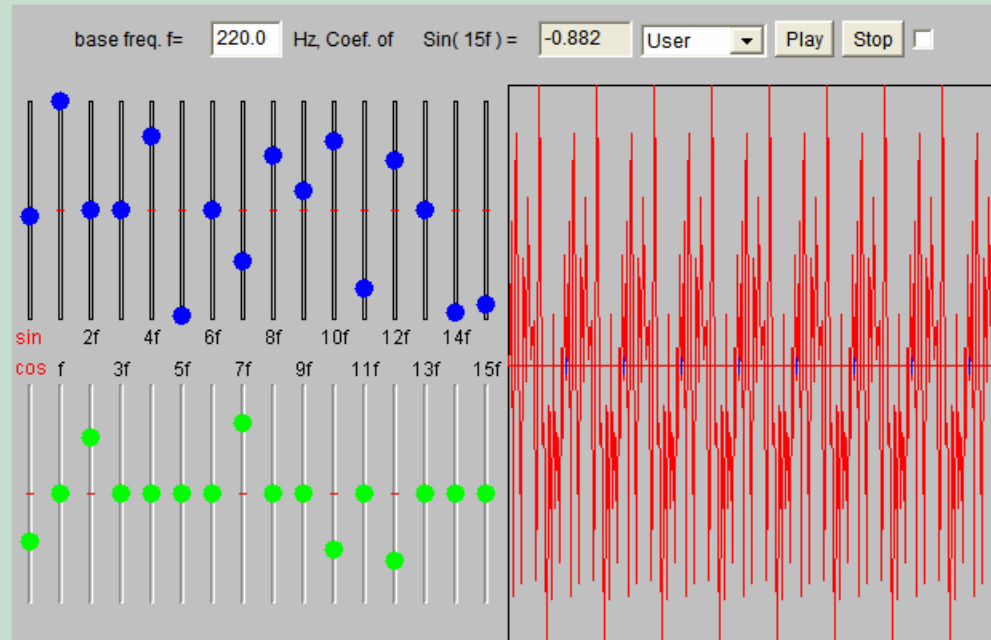
For coefficients that go to 0 fast enough these sums will converge at each value of  $t$ .

This defines a new function, which must be a *periodic function*.  
(Period  $2\pi$ )

Fourier's claim: *ANY* periodic function  $f(t)$  can be written this way

# Music

<http://www.phy.ntnu.edu.tw/ntnujava/viewton>



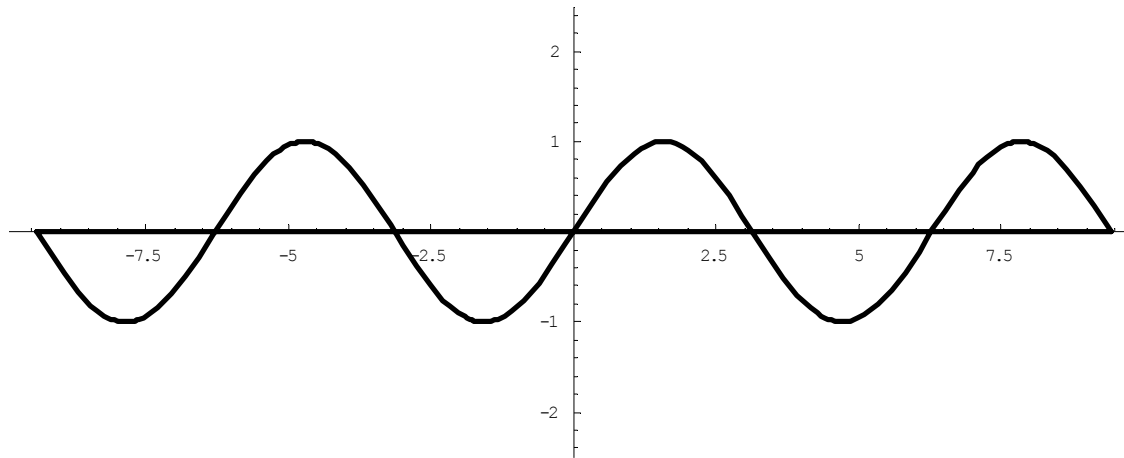
Click **get** button to fetch coefficients. Click **set** button to modify coefficients.

<input type="button" value="set"/>	0f	f	2f	3f	4f	5f	6f	7f
sin	<input type="button" value="get"/>	<input type="text" value="1.0"/>	<input type="text" value="0.0"/>	<input type="text" value="0.0"/>	<input type="text" value="0.69"/>	<input type="text" value="-0.98"/>	<input type="text" value="0.0"/>	<input type="text" value="-0.44"/>
cos	<input type="text" value="-0.44"/>	<input type="text" value="0.0"/>	<input type="text" value="0.52"/>	<input type="text" value="0.0"/>	<input type="text" value="0.0"/>	<input type="text" value="0.0"/>	<input type="text" value="0.0"/>	<input type="text" value="0.66"/>
	8f	9f	10f	11f	12f	13f	14f	15f
sin	<input type="text" value="0.51"/>	<input type="text" value="0.19"/>	<input type="text" value="0.64"/>	<input type="text" value="-0.73"/>	<input type="text" value="0.47"/>	<input type="text" value="0.0"/>	<input type="text" value="-0.93"/>	<input type="text" value="-0.88"/>

# Fourier's Representation

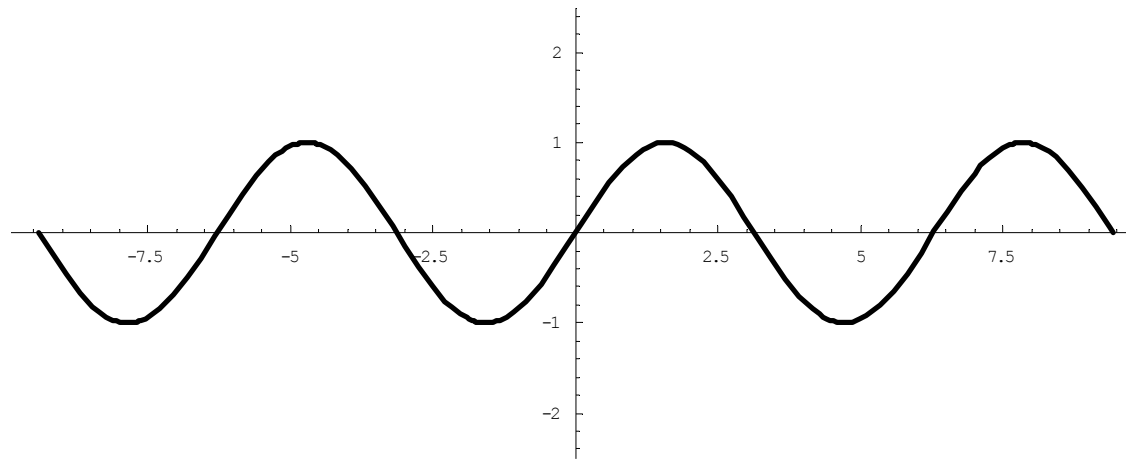
Represent  $f(t) = t$  for  $|t| < \pi$  and  $2\pi$  periodic

$\sin(t) + \dots$



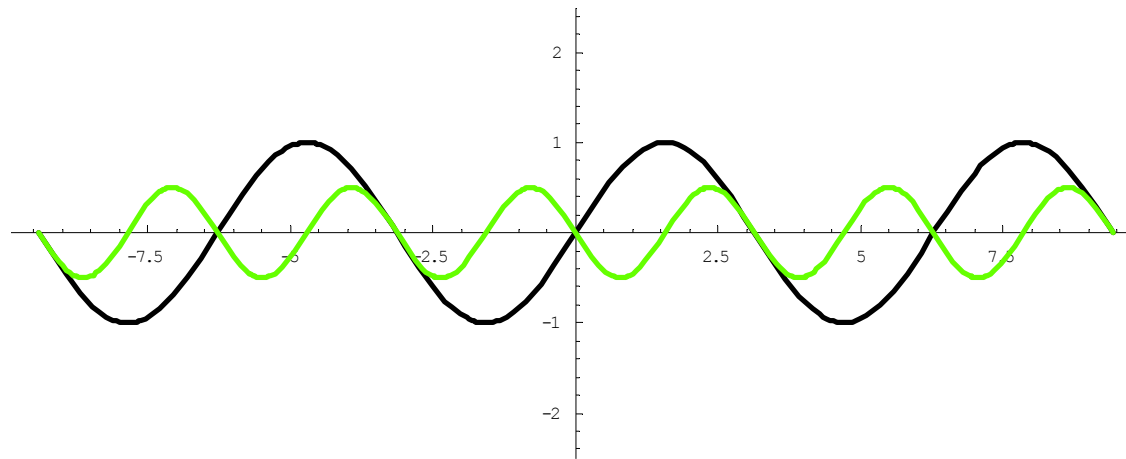
# Fourier's Representation

$1 \sin(t) + \dots$



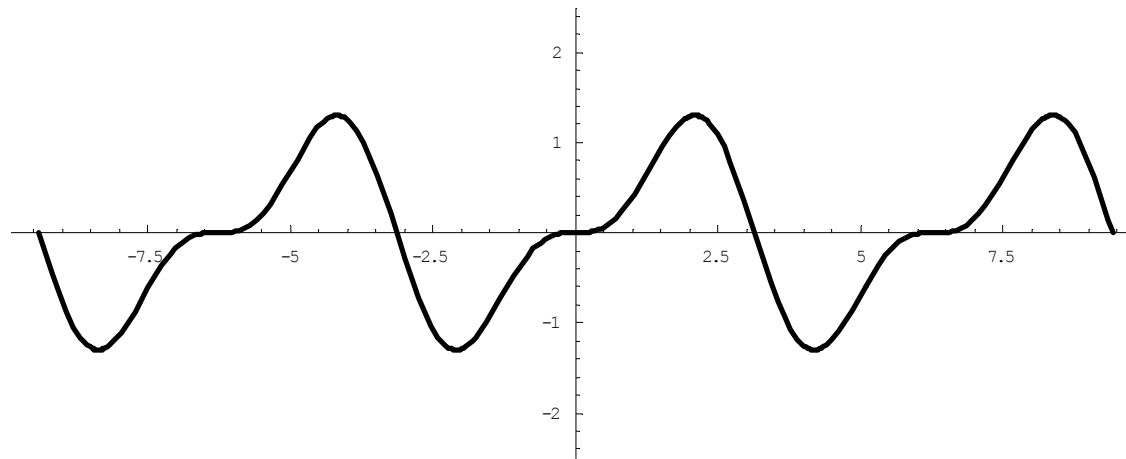
# Fourier's Representation

$$\sin(t) - \frac{1}{2} \sin(2t) + \dots$$



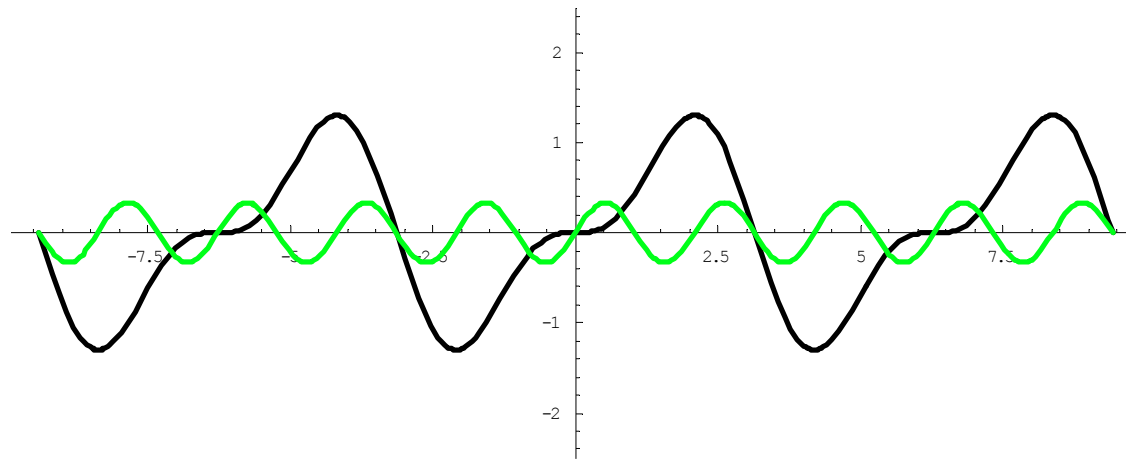
# Fourier's Representation

$$1 \sin(t) - \frac{1}{2} \sin(2t) + \dots$$



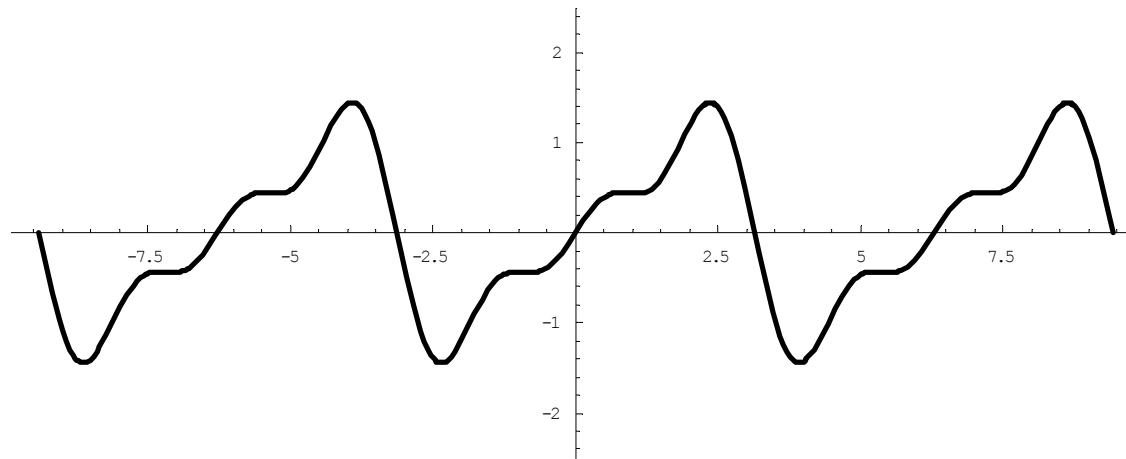
# Fourier's Representation

$$1 \sin(t) - \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \dots$$



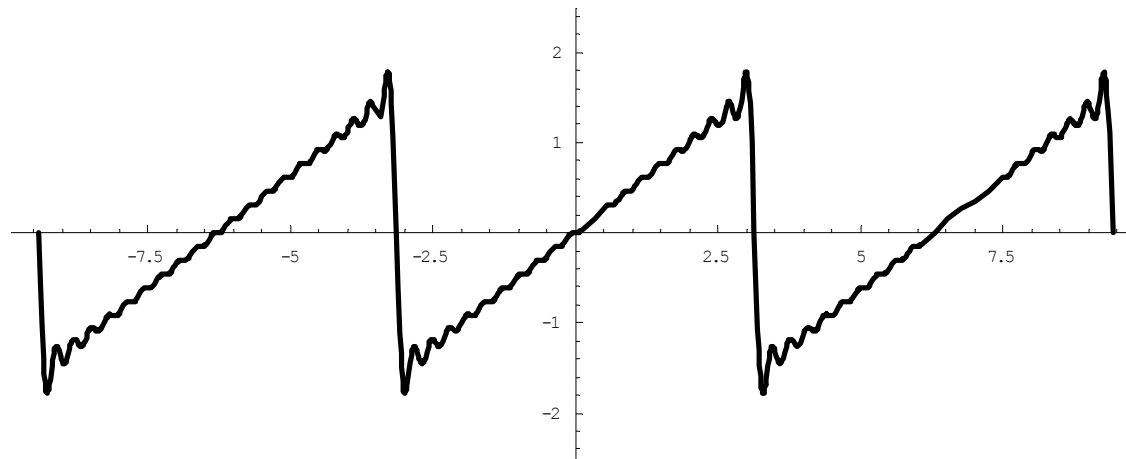
# Fourier's Representation

$$1 \sin(t) - \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \dots$$



# Fourier's Representation

20'th degree Fourier expansion



# How do you get the coefficients for a given $f$ ?

$$A_0/2 + A_1 \cos(t) + A_2 \cos(2t) + A_3 \cos(3t) + \dots \\ + B_1 \sin(t) + B_2 \sin(2t) + B_3 \sin(3t) + \dots$$

Fourier's claim:

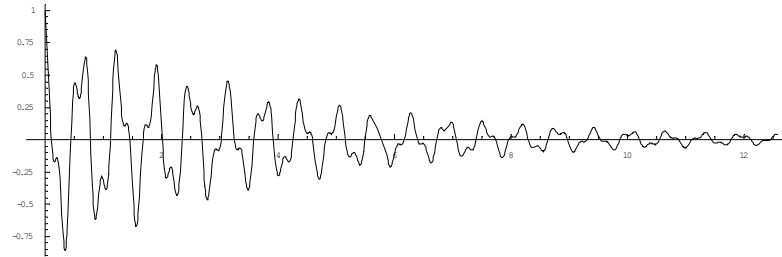
- *ANY* periodic function  $f(t)$  can be written this way (SYNTHESIS)
- The coefficients are uniquely determined by  $f$ : (ANALYSIS)

$$A_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cos(kt) dt$$

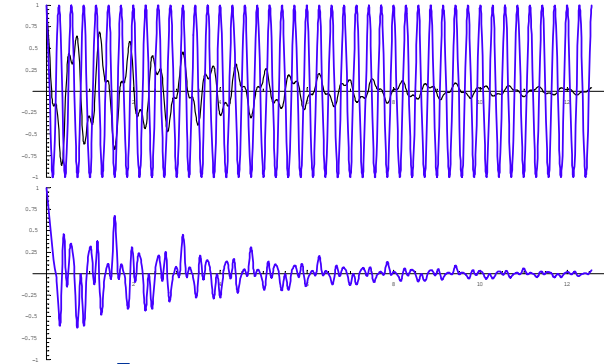
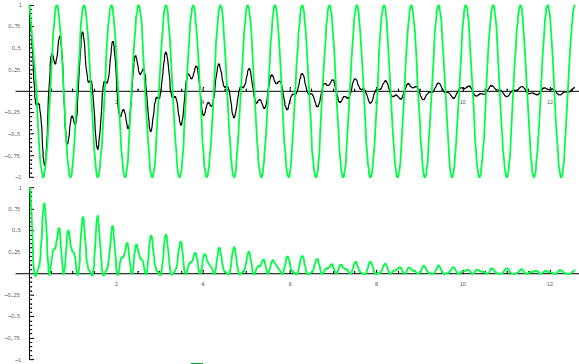
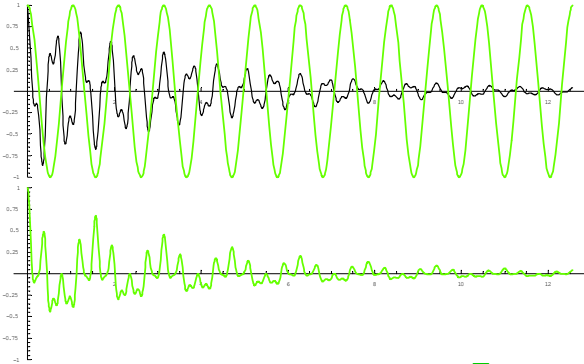
$$B_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \sin(kt) dt$$

# Fourier Analysis: match data with sinusoids

$s(t)$

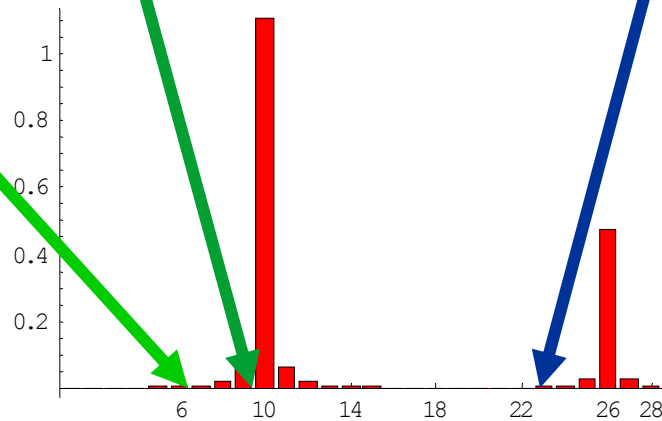


Time t



$S[k]$

$$= \int s(t) \cos(k t) dt$$



Frequency k

# Complex Notation

For  $f$ , periodic with period  $p$

Fourier transform  $f(t) \rightarrow F[k]$

$$\begin{aligned} F[k] &= \frac{1}{p} \int_0^p f(t) e^{-2\pi i k t/p} dt \\ &= \frac{1}{p} \int_0^p f(t) \cos(2\pi k t/p) dt \\ &\quad - \frac{i}{p} \int_0^p f(t) \sin(2\pi k t/p) dt \end{aligned}$$

Inverse Fourier transform  $F[k] \rightarrow f(t)$

$$f(t) = \sum_{k \in \mathbb{Z}} F[k] e^{2\pi i k t/p}$$

# Sampling

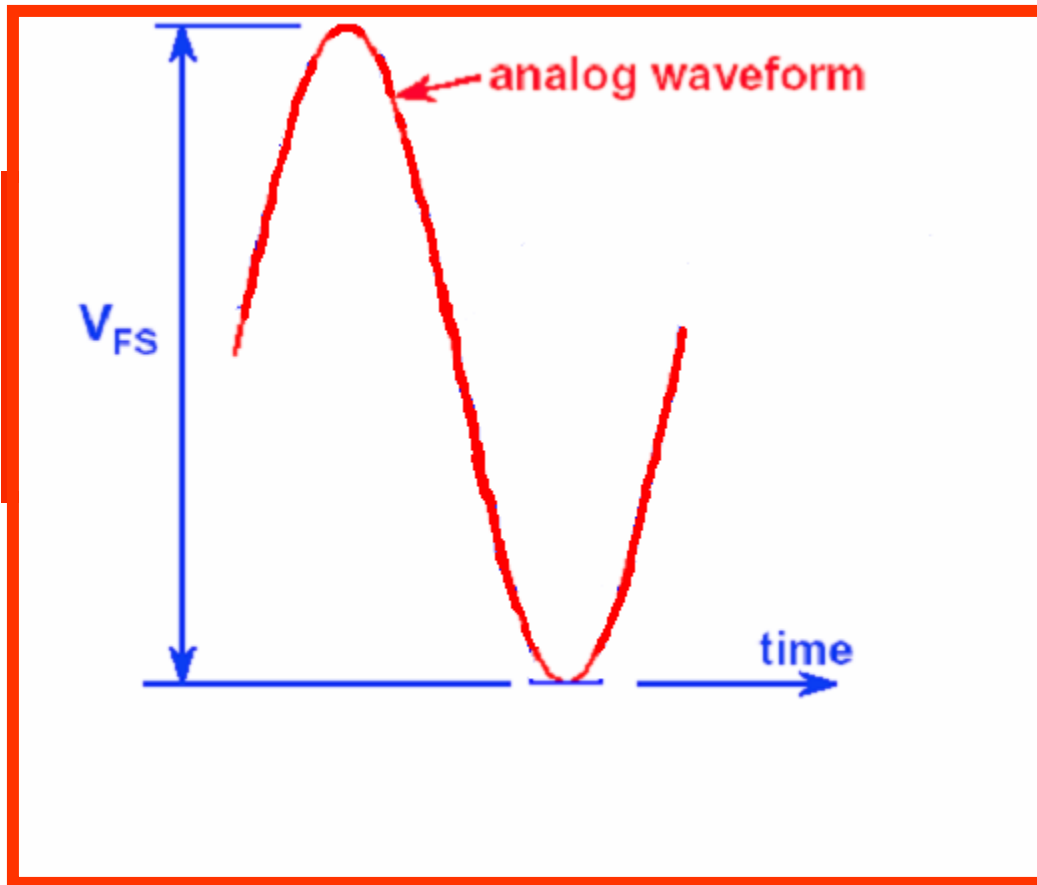
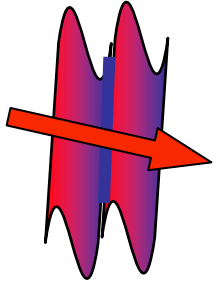
Fourier representations work just fine with sampled data

Simple connection to Fourier of the continuous function it came from

Familiar example: Digital Audio

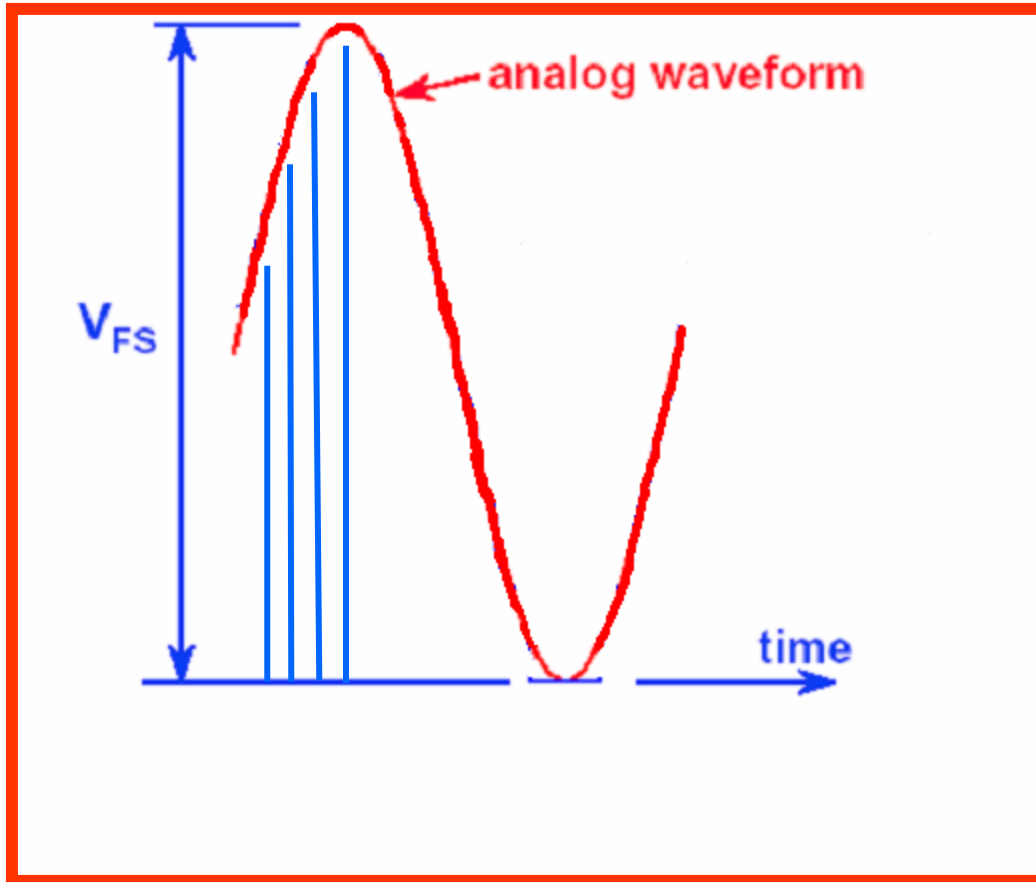
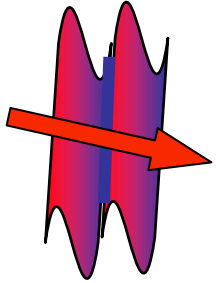
# Measuring and Discretizing Input field

Physical Field  
(continuum)



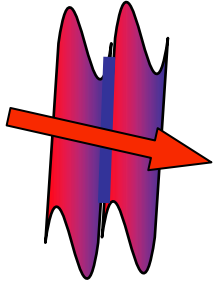
# Sample

Physical Field  
(continuum)

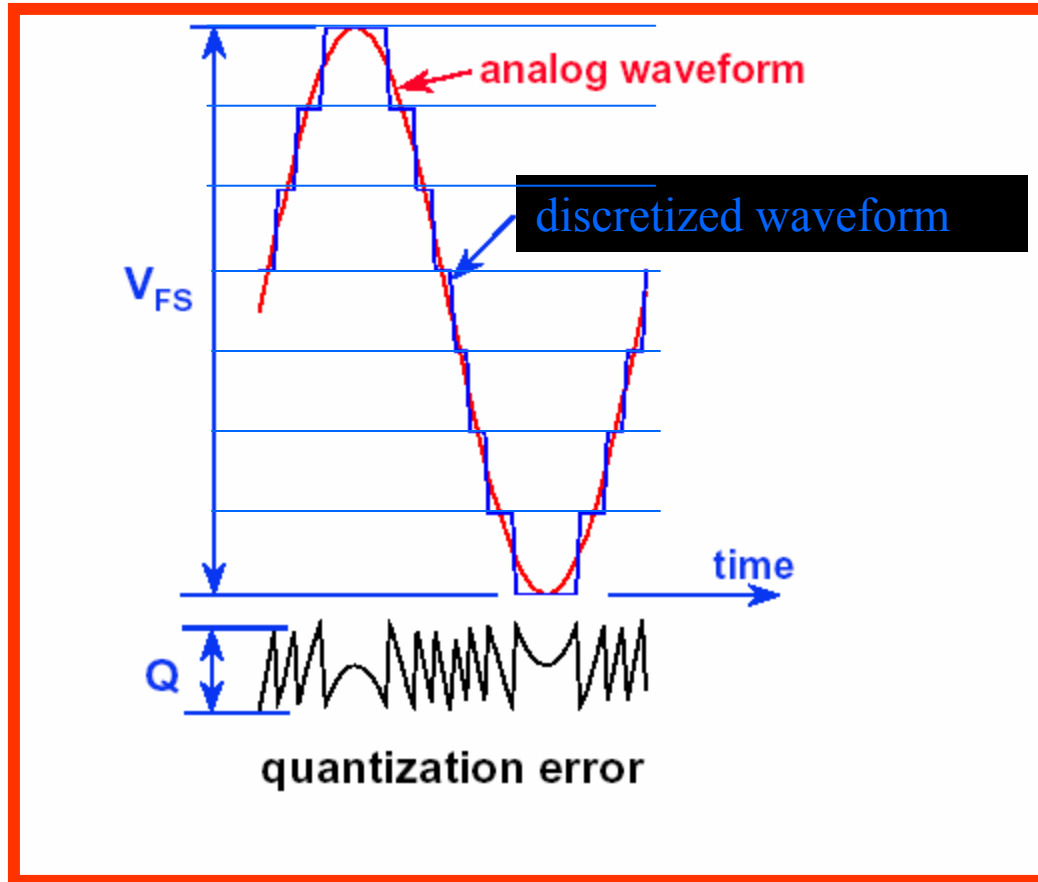


# Quantize

Physical Field  
(continuum)

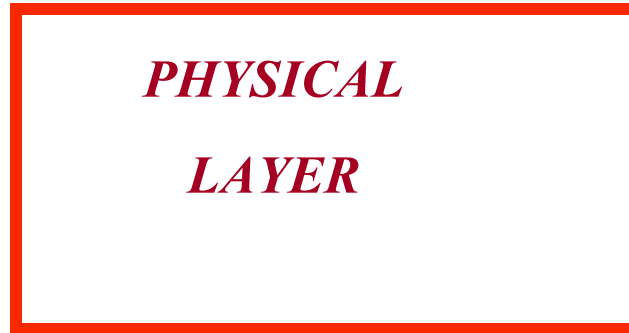
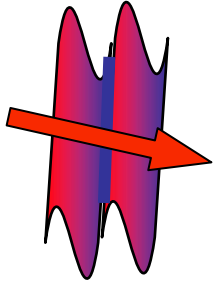


$$NP_0(rms) = \frac{Q}{\sqrt{12}}$$

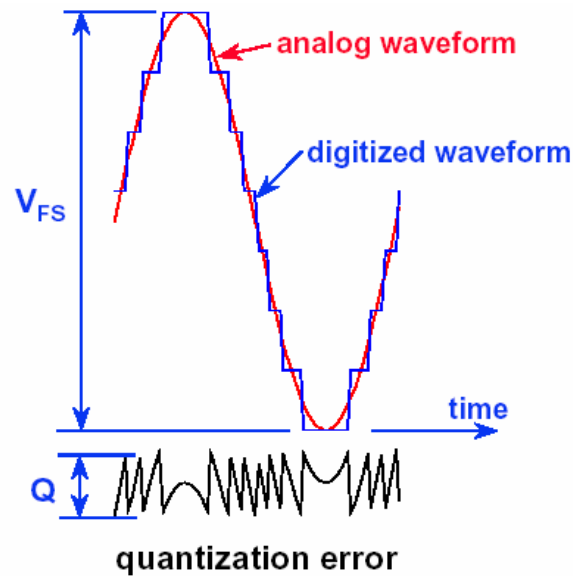
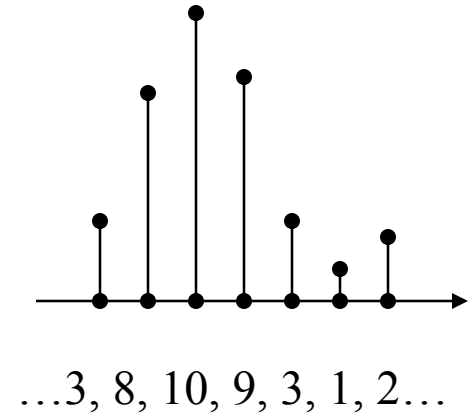


# Code and output

Physical Field  
(continuum)

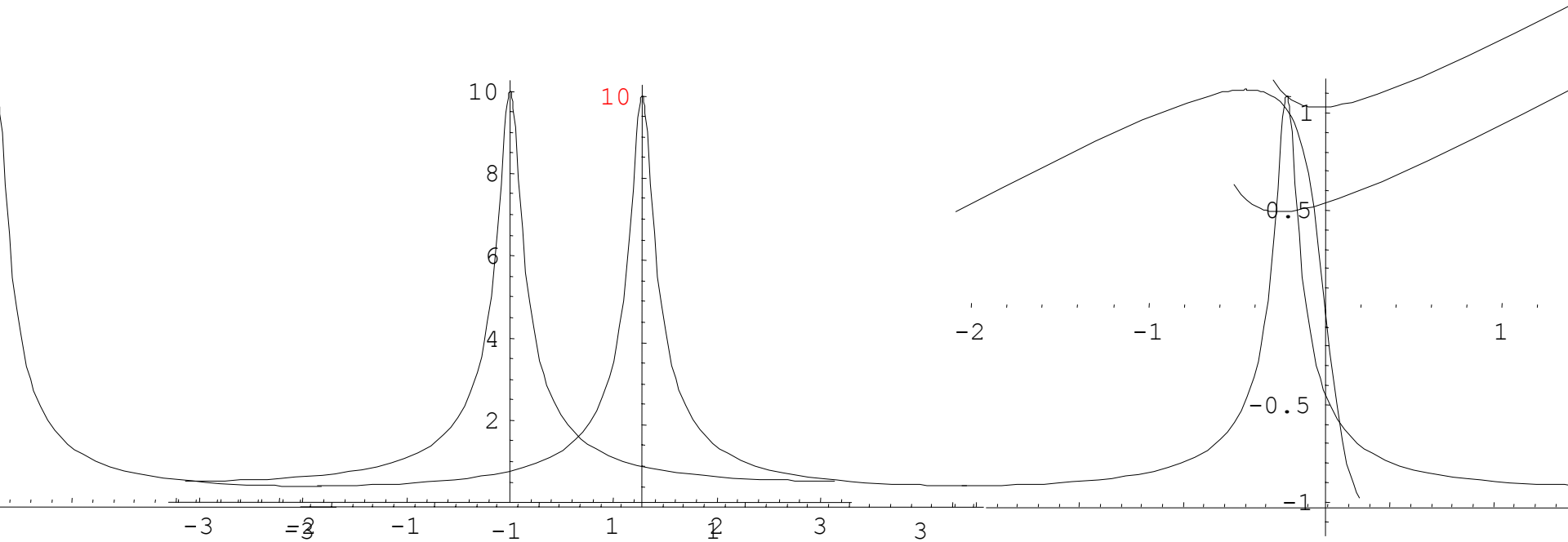


Digital Representation



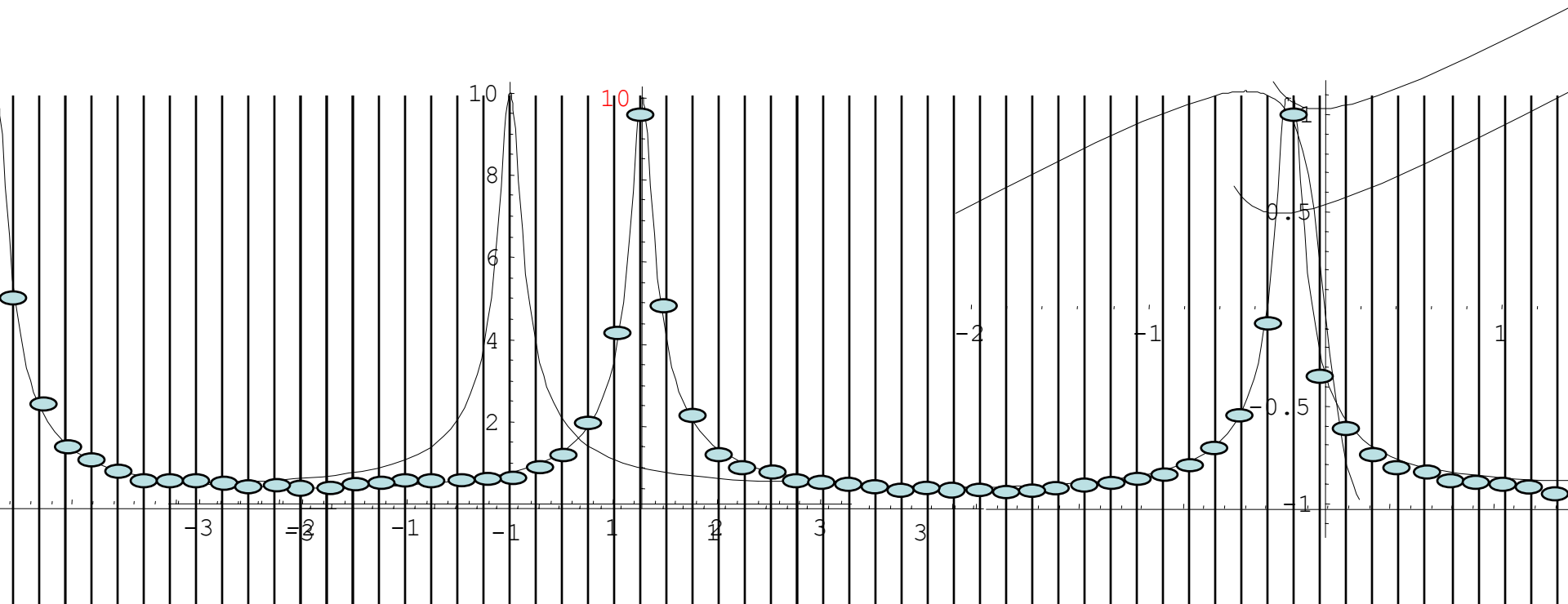
# Sampling

Often must work with a discrete set of measurements of a continuous function



# Sampling

Takes a function defined on  $\mathcal{R}$  and creates a function defined on  $\mathcal{Z}$



$S_h$

$f(t)$



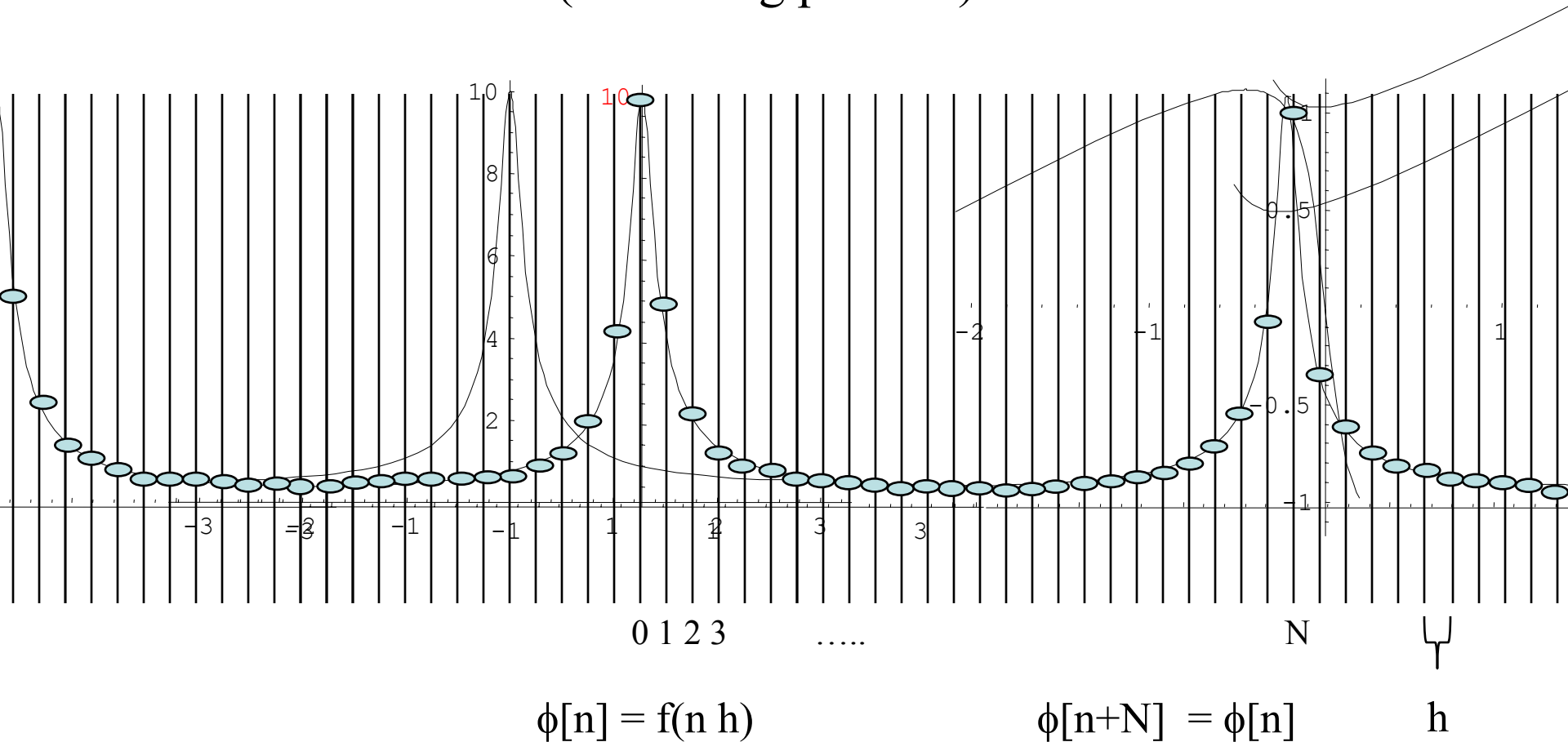
$\phi[n] = f(nh)$

$h$

# Sampling

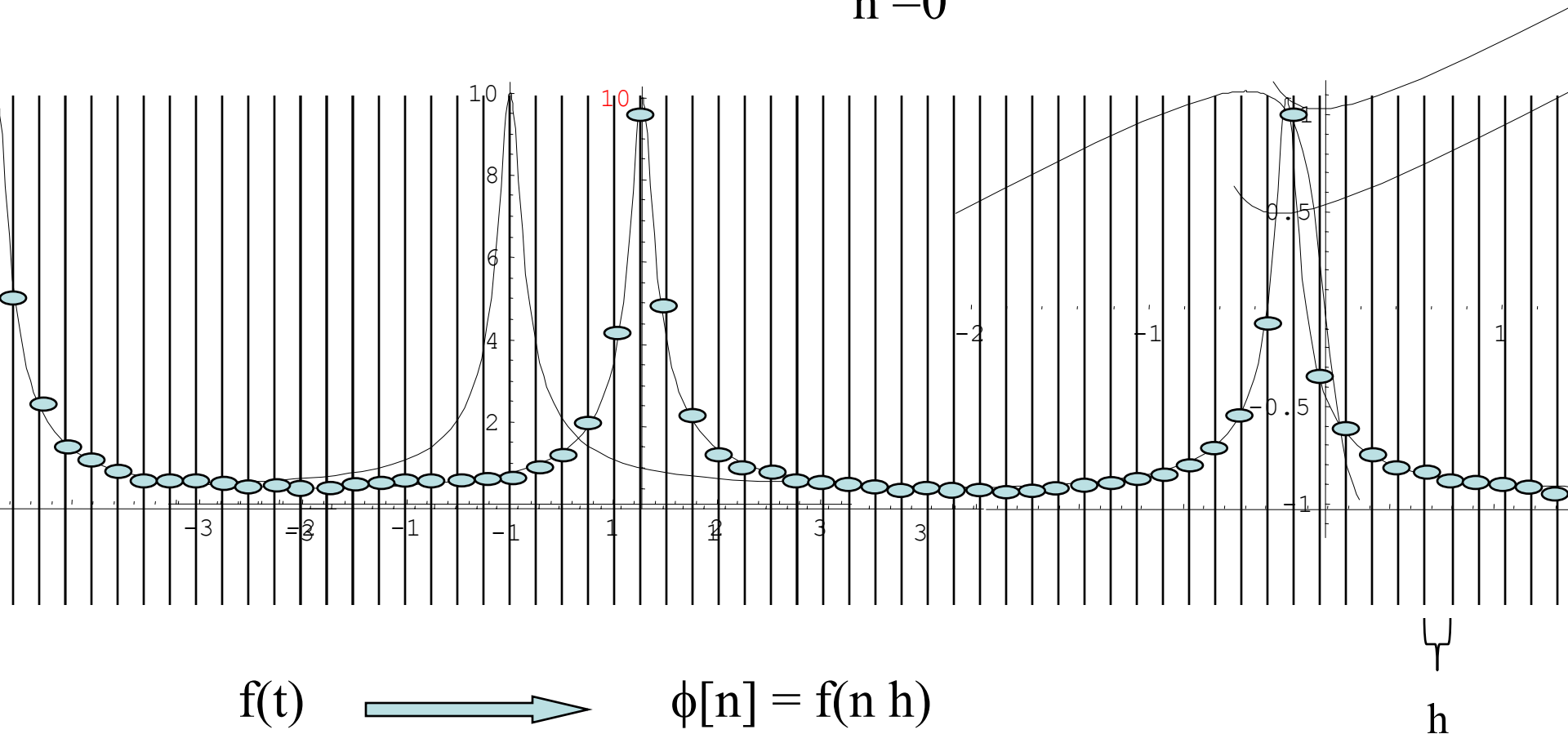
In this case, it is a periodic function on  $\mathbb{Z}$ ,

(Assuming  $p/h = N$ )



# DFT

$$\int_0^p f(t) e^{-2\pi i k t/p} dt \longrightarrow \sum_{n=0}^{N-1} \phi[n] e^{-2\pi i k n h/p}$$



# DFT and its inverse for periodic discrete data

$$\begin{aligned}\Phi[k] &= \sum_{n=0}^{N-1} \phi[n] e^{-2 \pi i k n h / p} & p &= N h \\ &= \sum_{n=0}^{N-1} \phi[n] e^{-2 \pi i k n / N}\end{aligned}$$

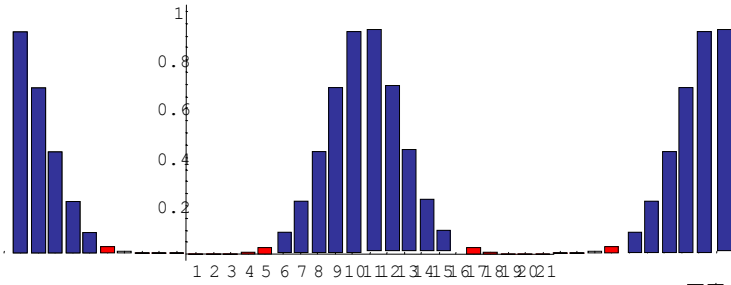
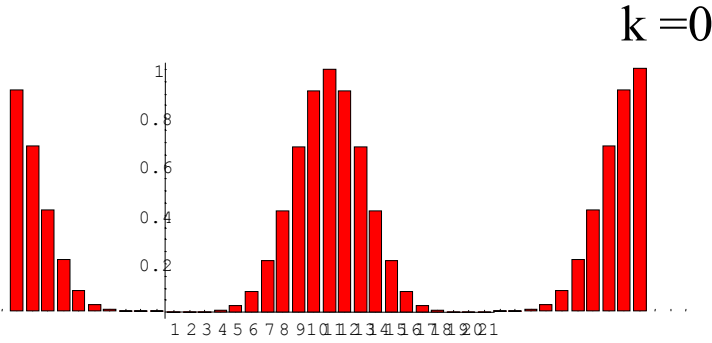
This is automatically periodic in  $k$  with period  $N$   
Inverse is like Fourier series, but with only  $p$  terms

# Discrete-time periodic version of Fourier

“time” domain

“frequency” domain

$$\frac{1}{N} \sum_{k=0}^{N-1} \gamma[k] e^{-2 \pi i k m/N} = \Gamma [m]$$

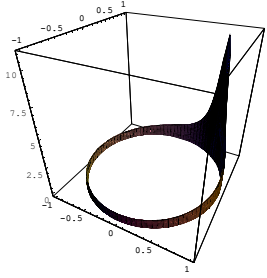


$$\gamma[k] = \sum_{m=0}^{N-1} \Gamma [m] e^{2 \pi i m k/N}$$

$\gamma[k]$ , on  $P_N$   
i.e. on  $\mathbb{Z}$ , Period N

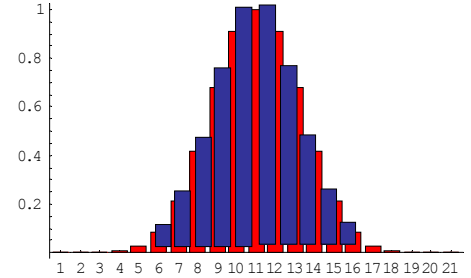
$\Gamma [m]$  on  $P_N$   
i.e. on  $\mathbb{Z}$ , Period N

# Fourier



$$\int_0^p f(t) e^{-2\pi i k t/p} dt / p$$

$$\sum_{k \in \mathcal{Z}} F[k] e^{2\pi i k t/p}$$



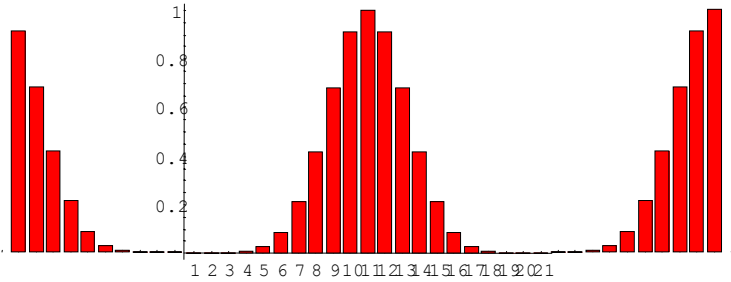
$f(t)$ , period  $p$

$F[k]$ , on  $\mathcal{Z}$

$\downarrow S_{p/N}$

$$1/N \sum_{k=0}^{N-1} \gamma[k] e^{-2\pi i k m/N}$$

$\downarrow P_N$



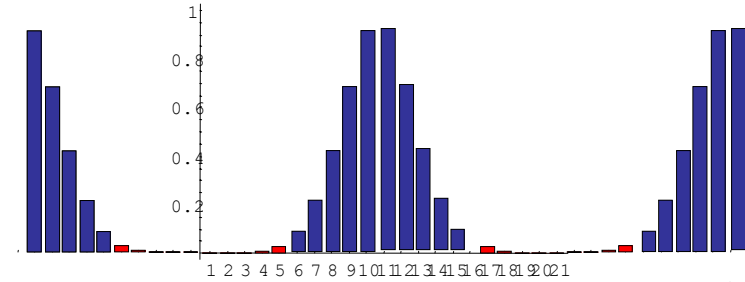
$\gamma[k]$ , on  $P_N$

i.e. on  $\mathcal{Z}$ , Period  $N$

$k=0$

$$\sum_{k=0}^{N-1} \Gamma[k] e^{2\pi i m k/N}$$

$k=0$



$\Gamma[m]$  on  $P_N$

i.e. on  $\mathcal{Z}$ , Period  $N$

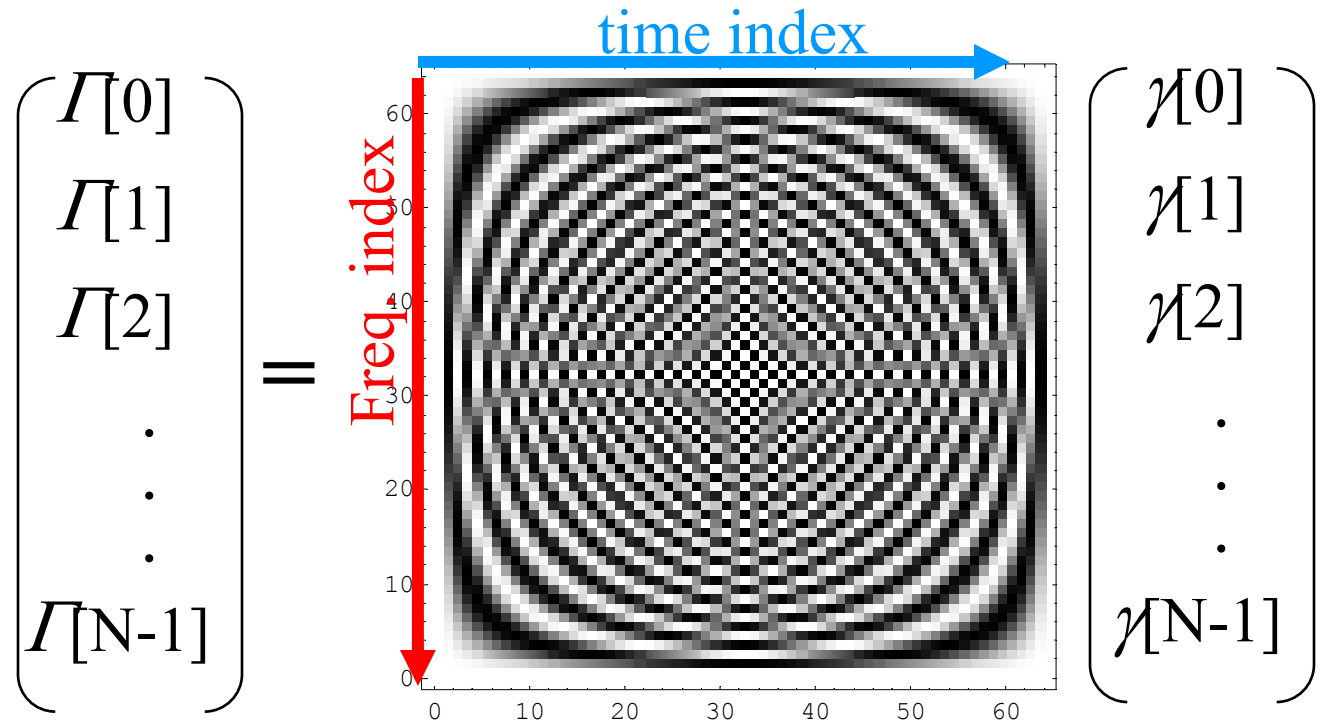
*“time” domain*

*“frequency” domain*

# Discrete time Numerical Fourier Analysis

DFT is really just a matrix multiplication!

$$\Gamma[m] = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i k m/N} \mathcal{H}[k]$$



$$\Gamma = F_N \mathcal{H}$$

# Numerical Harmonic Analysis

FFT: Symmetry Properties permits “Divide and Conquer”  
Sparse Factorization

$$F_{mn} = (F_m \otimes I_n) \cdot T_n^{mn} \cdot (I_m \otimes F_n) \cdot L_m^{mn}$$

