Separation of Variables

Some slides adapted from Ake Nordlund and Dennis Healy
Separation of variables …

• Main principles
  – Why?

• Techniques
  – How?
Main principles

• Why?
  – Because many systems in physics are separable
  – In particular systems with symmetries
  – Such as spherically symmetric problems

Example: Spherical harmonics

\[ u(r, \Theta, \phi, t) = Y_{lm}(r, \Theta, \phi) \, e^{i\omega t} \]
Partial Differential Equations: Separation of Variables …

• Main principles
  – Why?
    • Common in physics!
    • Greatly simplifies things!

• Techniques
  – How?
Separation of variables: the general method

• Suppose we seek a PDE solution $u(x,y,z,t)$

Ansatz:

$$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\frac{\partial u}{\partial x} = X'(x) Y(y) Z(z) T(t)$$
$$\frac{\partial^2 u}{\partial x^2} = X''(x) Y(y) Z(z) T(t)$$
Separation of variables: the general method

• Suppose we seek a PDE solution $u(x,y,z,t)$:

   Ansatz:
   $$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

   This ansatz may be correct or incorrect!

   If correct we have

   $$\frac{\partial u}{\partial y} = X(x) Y'(y) Z(z) T(t)$$
   $$\frac{\partial^2 u}{\partial y^2} = X(x) Y''(y) Z(z) T(t)$$
Separation of variables: the general method

• Suppose we seek a PDE solution \( u(x,y,z,t) \):

   **Ansatz:**
   \[
   u(x,y,z,t) = X(x) \ Y(y) \ Z(z) \ T(t)
   \]

   This ansatz may be correct or incorrect!

   If correct we have

   \[
   \frac{\partial u}{\partial z} = X(x) \ Y(y) \ Z'(z) \ T(t)
   \]

   \[
   \frac{\partial^2 u}{\partial z^2} = X(x) \ Y(y) \ Z''(z) \ T(t)
   \]
Separation of variables: the general method

- Suppose we seek a PDE solution $u(x,y,z,t)$:

  Ansatz:
  $$u(x,y,z,t) = X(x) Y(y) Z(z) T(t)$$

  This ansatz may be correct or incorrect!
  If correct we have
  $$\frac{\partial u}{\partial t} = X(x) Y(y) Z(z) T'(t)$$
  $$\frac{\partial^2 u}{\partial t^2} = X(x) Y(y) Z(z) T''(t)$$
Examples

\[ u(x,y,z,t) = xyz^2 \sin(bt) \]

\[ u(x,y,z,t) = xy + zt \]

\[ u(x,y,z,t) = (x^2+y^2) z \cos(\omega t) \]
Examples

Separable?

Yes!

No!

Hm…
Examples

Yes!

No!

Hm…
Examples

Separable?

- $u(x,y,z,t) = xyz^2 \sin(bt)$  
  Yes!

- $u(x,y,z,t) = xy + zt$  
  No!

- $u(p,z,t) = p^2 z \cos(\omega t)$  
  Yes!
Example PDEs

• The wave equation

\[ c^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2} \]

where

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

Ansatz:

\[ u(x,y,z,t) = X(x) Y(y) Z(z) T(t) \]
Separation of variables in the wave equation

• For the ansatz to work we must have (let's set $c=1$ for now to get rid of a triviality!)

\[ X''YZT + XY''ZT + XYZ''T = XYZT'' \]

• Divide though with $XYZT$, for:

\[ X''/X + Y''/Y + Z''/Z = T''/T \]

This can only work if all of these are constants!
Separation of variables in the wave equation

• Let’s take all the constants to be negative
  – Negative ⇒ bounded behavior at large x,y,z,t

X’’/X + Y’’/Y + Z’’/Z = T’’/T

These are called separation constants

- ⚫² - ○² - □² = - µ²

This is typical: If / when a PDE allows separation of variables, the partial derivatives are replaced with ordinary derivatives, and all that remains of the PDE is an algebraic equation and a set of ODEs – much easier to solve!
Separation of variables in the wave equation

• Solutions of the ODEs:

Example: x-direction

\[ \frac{X''}{X} = -\omega^2 \]

General solution:

\[ X(x) = A \exp(i\omega x) + B \exp(-i\omega x) \]

or

\[ X(x) = A' \cos(\omega x) + B' \sin(\omega x) \]
Separation of variables in the wave equation

• Combining all directions (and time)

Example:

\[ u = \exp(i \bullet x) \exp(i \bigcirc y) \exp(i \blacklozenge z) \exp(-i c \mu t) \]

or

\[ u = \exp(i (\bullet x + \bigcirc y + \blacklozenge z - \omega t)) \]

This is a traveling wave, with wave vector \( \{\bullet, \bigcirc, \blacklozenge\} \) and frequency \( \omega \). A general solution of the wave equation is a super-position of such waves.
Example PDEs

• The **Laplace equation**

\[ \kappa \nabla^2 u = 0 \]

where

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

Ansatz:

\[ u(x,y,z) = X(x) Y(y) Z(z) \]
Separation of variables in the Laplace equation

• For the ansatz to work we must have

\[ X''YZ + XY''Z + XYZ''' = 0 \]

• Divide though with \( XYZ \), for:

\[ \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \]

Again, this can only work if all of these are constants!
Separation of variables in the Laplace equation

- This time we cannot take all constants to be negative!

\[
\frac{X''}{X} + \frac{Y''}{Y} = 0
\]

\[
- \bullet^2 - \bigcirc^2 = 0
\]

At least one of the terms must be positive \(\Rightarrow\) imaginary wave number; i.e., **exponential** rather than sinusoidal behavior!
Separation of variables in the Laplace equation

• Solutions of the ODEs:

Example: x-direction

\[ \frac{X''}{X} = -\bullet^2 < 0 \text{ assumed} \]

General solution:

\[ X(x) = A \exp(i \bullet x) + B \exp(-i \bullet x) \]

or

\[ X(x) = A' \cos(\bullet x) + B' \sin(\bullet x) \]
Separation of variables in the Laplace equation

- Solutions of the ODEs:

Example: y-direction

\[ \frac{Y''}{Y} = + \cdot \cdot^2 \quad > 0 \quad \text{follows!} \]

General solution:

\[ Y(y) = C \exp(\cdot \cdot y) + D \exp(-\cdot \cdot y) \]

or

\[ Y(y) = C' \cosh(\cdot \cdot y) + B' \sinh(\cdot \cdot y) \]
Separation of variables in the Laplace equation

• Combining x & y directions

Example:

\[ u = [A \exp(i \bullet x) + B \exp(-i \bullet x)] \ [C \exp(\bullet y) + D \exp(-\bullet y)] \]

or

\[ u = [A' \cos(\bullet x) + B' \sin(\bullet x)] \ [C' \cosh(\bullet y) + D' \sinh(-\bullet y)] \]
Message

• Complex exponentials are a natural basis for representing the solutions of the wave equation
• A microphone samples the solution at a given point
• Variation in time is naturally expressed in terms of Fourier series.
Fourier Methods

• Fourier analysis (“harmonic analysis”) a key field of math
• Has many applications and has enabled many technologies.
• Basic idea: Use Fourier representation to represent functions.
• Has fast algorithms to manipulate them (the fast Fourier Transform)
Basic idea

• Function spaces can have many different types of bases
• We have already met monomials and other polynomial basis functions
• Fourier introduced another set of basis functions: the Fourier series
• These basis functions are particularly good for describing things that repeat with time
Fourier’s Representation

\[ F(t) = A_0/2 + A_1 \cos(t) + A_2 \cos(2t) + A_3 \cos(3t) + \ldots \\
+ B_1 \sin(t) + B_2 \sin(2t) + B_3 \sin(3t) + \ldots \]

For coefficients that go to 0 fast enough these sums will converge at each value of \( t \).

This defines a new function, which must be a periodic function. (Period \( 2\pi \))

Fourier’s claim: ANY periodic function \( f(t) \) can be written this way
Music

http://www.phy.ntnu.edu.tw/ntnujava/viewtopic.php?t=33
Fourier’s Representation

Represent $f(t) = t$ for $|t| < \pi$ and $2\pi$ periodic

$\sin(t) + \ldots$
Fourier’s Representation

$1 \sin(t) + \ldots$
Fourier’s Representation

\[ \sin(t) - \frac{1}{2} \sin(2t) + \ldots \]
Fourier’s Representation

$1 \sin(t) - \frac{1}{2} \sin(2t) + \ldots$
Fourier’s Representation

\[ 1 \sin(t) - \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \ldots \]
Fourier’s Representation

\[ 1 \sin(t) - \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \ldots \]
Fourier’s Representation

20’th degree Fourier expansion
How do you get the Coefficients for a given $f$?

$$A_0/2 + A_1 \cos(t) + A_2 \cos(2t) + A_3 \cos(3t) + \ldots$$

$$+ B_1 \sin(t) + B_2 \sin(2t) + B_3 \sin(3t) + \ldots$$

Fourier’s claim:

• *ANY* periodic function $f(t)$ can be written this way (SYNTHESIS)

• The coefficients are uniquely determined by $f$: (ANALYSIS)

$$A_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cos(kt) \, dt$$

$$B_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \sin(kt) \, dt$$
Fourier Analysis: match data with sinusoids

\[ S[k] = \int s(t) \cos(k t) \, dt \]
Complex Notation

For \( f \), periodic with period \( p \)

**Fourier transform**  \( f(t) \rightarrow F[k] \)

\[
F[k] = \frac{1}{p} \int_{0}^{p} f(t) e^{-2 \pi i \frac{k}{p} t} \, dt
\]

**Inverse Fourier transform**  \( F[k] \rightarrow f(t) \)

\[
f(t) = \sum_{k \text{ in } \mathbb{Z}} F[k] e^{2 \pi i \frac{k}{p} t}
\]
Sampling

Fourier representations work just fine with sampled data

Simple connection to Fourier of the continuous function it came from

Familiar example: Digital Audio
Measuring and Discretizing Input field

Physical Field
(continuum)

$V_{FS}$

analog waveform

time
Physical Field
(continuum)

$V_{FS}$

analog waveform

time
Quantize

Physical Field (continuum)

\[ NP_0(rms) = \frac{Q}{\sqrt{12}} \]
Code and output

Physical Field (continuum)

**PHYSICAL LAYER**

Digital Representation

...3, 8, 10, 9, 3, 1, 2...

![Diagram showing physical field and digital representation](image)
Sampling

Often must work with a discrete set of measurements of a continuous function
Sampling

Takes a function defined on $\mathbb{R}$ and creates a function defined on $\mathbb{Z}$. 

$S_h f(t) \rightarrow \phi[n] = f(n h)$
Sampling

In this case, it is a periodic function on $\mathbb{Z}$,

(Assuming $p/h = N$)

\[\phi[n] = f(n \cdot h) \quad \phi[n+N] = \phi[n]\]
DFT

\[ \int_{0}^{p} f(t) \ e^{-2\pi\frac{ikt}{p}} \ dt \rightarrow \sum_{n=0}^{N-1} \phi[n] \ e^{-2\pi\frac{iknh}{p}} \]

\[ f(t) \rightarrow \phi[n] = f(nh) \]
DFT and its inverse for periodic discrete data

\[ \Phi[k] = \sum_{n=0}^{N-1} \phi[n] e^{-2\pi i k n h/p} \quad p = N h \]

\[ = \sum_{n=0}^{N-1} \phi[n] e^{-2\pi i k n / N} \]

This is automatically periodic in \( k \) with period \( N \).

Inverse is like Fourier series, but with only \( p \) terms.
DTFT: Discrete time periodic version of Fourier

"time" domain

\[ \gamma[k] = \sum_{k=0}^{N-1} \frac{1}{N} \sum_{m} \gamma[k] e^{-2\pi i km/N} = \Gamma[m] \]

\[ \gamma[k] = \sum_{k=0}^{N-1} \Gamma[k] e^{2\pi i km/N} \]

\( \gamma[k] \), on \( \mathcal{P}_N \)

i.e. on \( \mathbb{Z} \), Period \( N \)

"frequency" domain

\( \Gamma[m] \) on \( \mathcal{P}_N \)

i.e. on \( \mathbb{Z} \), Period \( N \)
Fourier

\[ \int_0^p f(t) \ e^{-2 \pi i k t/p} \ dt /p \]

\[ \sum_{k \in \mathbb{Z}} F[k] \ e^{2 \pi i k t/p} \]

\[ \frac{1}{N} \sum_{k = 0}^{N-1} \gamma[k] \ e^{-2 \pi i k m/N} \]

\[ \sum_{k = 0}^{N-1} \Gamma[k] \ e^{2 \pi i m k/N} \]

\( f(t), \text{ period } p \)

\( F[k], \text{ on } \mathbb{Z} \)

\( \gamma[k], \text{ on } \mathbb{P}_N \)

\( \Gamma[m], \text{ on } \mathbb{P}_N \)

\( \text{“time” domain} \)

\( \text{“frequency” domain} \)
Discrete time Numerical Fourier Analysis

DFT is really just a matrix multiplication!

\[ \Gamma \begin{bmatrix} \gamma[0] \\ \gamma[1] \\ \gamma[2] \\ \vdots \\ \gamma[N-1] \end{bmatrix} = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i k m/N} \gamma[k] \]

\[ \begin{bmatrix} \gamma[0] \\ \gamma[1] \\ \gamma[2] \\ \vdots \\ \gamma[N-1] \end{bmatrix} = F_N \begin{bmatrix} \gamma[0] \\ \gamma[1] \\ \gamma[2] \\ \vdots \\ \gamma[N-1] \end{bmatrix} \]
Numerical Harmonic Analysis

FFT: Symmetry Properties permits "Divide and Conquer"
Sparse Factorization

\[ F_{mn} = (F_m \otimes I_n) \cdot T_{mn} \cdot (I_m \otimes F_n) \cdot L_{mn} \]

Naive

\[ F_n \]

FFT