Separation of Variables

Some slides adapted from Ake Nordlund and Dennis Healy

Separation of variables …

- Main principles
  - Why?
- Techniques
  - How?

Main principles

- Why?
  - Because many systems in physics are separable
  - In particular systems with symmetries
  - Such as spherically symmetric problems

Example: Spherical harmonics

$$u(r,\Theta,\varphi,t) = Y_{lm}(r,\Theta,\varphi)\, e^{i\omega t}$$

Partial Differential Equations: Separation of Variables …

- Main principles
  - Why?
    - Common in physics!
    - Greatly simplifies things!

- Techniques
  - How?

Separation of variables: the general method

- Suppose we seek a PDE solution u(x,y,z,t)

Ansatz:

$$u(x,y,z,t) = X(x)\, Y(y)\, Z(z)\, T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\partial u/\partial x = X'(x)\, Y(y)\, Z(z)\, T(t)$$

$$\partial^2 u/\partial x^2 = X''(x)\, Y(y)\, Z(z)\, T(t)$$

Separation of variables: the general method

- Suppose we seek a PDE solution $u(x,y,z,t)$:

Ansatz:

$$u(x,y,z,t) = X(x)\, Y(y)\, Z(z)\, T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\partial u/\partial y = X(x)\, Y'(y)\, Z(z)\, T(t)$$

$$\partial^2 u/\partial y^2 = X(x)\, Y''(y)\, Z(z)\, T(t)$$

Separation of variables: the general method

- Suppose we seek a PDE solution $u(x,y,z,t)$:

Ansatz:

$$u(x,y,z,t) = X(x)\, Y(y)\, Z(z)\, T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\partial u/\partial z = X(x)\, Y(y)\, Z'(z)\, T(t)$$

$$\partial^2 u/\partial z^2 = X(x)\, Y(y)\, Z''(z)\, T(t)$$

Separation of variables: the general method

- Suppose we seek a PDE solution $u(x,y,z,t)$:

Ansatz:

$$u(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,T(t)$$

This ansatz may be correct or incorrect!

If correct we have

$$\partial u/\partial t = X(x)\,Y(y)\,Z(z)\,T'(t)$$

$$\partial^2 u/\partial t^2 = X(x)\,Y(y)\,Z(z)\,T''(t)$$

Examples

$$u(x,y,z,t) = xyz^2\sin(bt)$$

$$u(x,y,z,t) = xy + zt$$

$$u(x,y,z,t) = (x^2+y^2)\, z \cos(\omega t)$$

Examples

Separable?

✓ $u(x,y,z,t) = xyz^2\sin(bt)$ Yes!

~~$u(x,y,z,t) = xy + zt$~~ No!

? $u(x,y,z,t) = (x^2+y^2)\,z\cos(\omega t)$ Hm…

Examples

Separable?

✓ $u(x,y,z,t) = xyz^2\sin(bt)$ Yes!

~~$u(x,y,z,t) = xy + zt$~~ No!

? $u(x,y,z,t) = (x^2+y^2)\, z \cos(\omega t)$ Hm…

$(x^2+y^2) \Rightarrow p^2$

Examples

Separable?

✓ $u(x,y,z,t) = xyz^2\sin(bt)$ Yes!

~~$u(x,y,z,t) = xy + zt$~~ No!

✓ $u(p,z,t) = p^2 \; z\cos(\omega t)$ Yes!

Example PDEs

- The wave equation

$$c^2\nabla^2 u = \partial^2 u/\partial t^2$$

where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

Ansatz:

$$u(x,y,z,t) = X(x)\, Y(y)\, Z(z)\, T(t)$$

Separation of variables in the wave equation

- For the ansatz to work we must have (lets set c=1 for now to get rid of a triviality!)

$$X''YZT + XY''ZT + XYZ''T = XYZT''$$

- Divide though with XYZT, for:

$$X''/X + Y''/Y + Z''/Z = T''/T$$

This can only work if all of these are constants!

Separation of variables in the wave equation

- Let's take all the constants to be negative
  - Negative $\Rightarrow$ bounded behavior at large x,y,z,t

$$X''/X + Y''/Y + Z''/Z = T''/T$$

These are called **separation constants**

$$- \bullet^2 \quad - \bigcirc^2 \quad - \blacksquare^2 \quad = - \mu^2$$

This is typical: If / when a PDE allows separation of variables, the partial derivatives are replaced with ordinary derivatives, and all that remains of the PDE is an algebraic equation and a set of ODEs – much easier to solve!

Separation of variables in the wave equation

- Solutions of the ODEs:

Example: x-direction

$$X''/X = - \bullet^2$$

General solution:

$$X(x) = A \exp(i \bullet x) + B \exp(-i \bullet x)$$

or

$$X(x) = A' \cos(\bullet x) + B' \sin(\bullet x)$$

Separation of variables in the wave equation

- Combining all directions (and time)

Example:

$$u = \exp(i \bullet x) \exp(i ❍ y) \exp(i ■ z) \exp(-i c\mu t)$$

or

$$u = \exp(i (\bullet x + ❍ y + ■ z - \omega t))$$

This is a traveling wave, with wave vector {●,❍,■} and frequency ω. A general solution of the wave equation is a super-position of such waves.

Example PDEs

- The Laplace equation

$$\kappa \nabla^2 u = 0$$

where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

Ansatz:

$$u(x,y,z) = X(x)\, Y(y)\, Z(z)$$

Separation of variables in the Laplace equation

- For the ansatz to work we must have

$$X''YZ + XY''Z + XYZ'' = 0$$

- Divide though with XYZ, for:

$$X''/X + Y''/Y + Z''/Z = 0$$

Again, this can only work if all of these are constants!

Separation of variables in the Laplace equation

- This time we cannot take all constants to be negative!

$$X''/X + Y''/Y = 0$$

$$- \bullet^2 \quad - \bigcirc^2 = 0$$

At least one of the terms must be positive $\Rightarrow$ imaginary wave number; i.e., **exponential** rather than sinusoidal behavior!

Separation of variables in the Laplace equation

- Solutions of the ODEs:

Example: x-direction

$X''/X = -\bullet^2 \quad < 0 \quad$ assumed

General solution:

$X(x) = A \exp(i \bullet x) + B \exp(-i \bullet x)$

or

$X(x) = A' \cos(\bullet x) + B' \sin(\bullet x)$

Separation of variables in the Laplace equation

- Solutions of the ODEs:

Example: y-direction

$Y''/Y = + \bullet^2 \quad > 0 \quad$ follows!

General solution:

$Y(y) = C \exp(\bullet\, y) + D \exp(-\bullet\, y)$

or

$Y(y) = C' \cosh(\bullet\, y) + B' \sinh(\bullet\, y)$

Separation of variables in the Laplace equation

- Combining x & y directions

Example:

$$u = [A \exp(i \bullet x) + B \exp(-i \bullet x)]\,[C \exp(\bullet y) + D \exp(-\bullet y)]$$

or

$$u = [A' \cos(\bullet x) + B' \sin(\bullet x)]\,[C' \cosh(\bullet y) + D' \sinh(-\bullet y)]$$

Message

- Complex exponentials are a natural basis for representing the solutions of the wave equation
- A microphone samples the solution at a given point
- Variation in time is naturally expressed in terms of Fourier series.

Fourier Methods

- Fourier analysis (“harmonic analysis”) a key field of math
- Has many applications and has enabled many technologies.
- Basic idea: Use Fourier representation to represent functions
- Has fast algorithms to manipulate them (the fast Fourier Transform)

Basic idea

- Function spaces can have many different types of bases
- We have already met monomials and other polynomial basis functions
- Fourier introduced another set of basis functions : the Fourier series
- These basis functions are particularly good for describing things that repeat with time

Fourier's Representation

$$F(t) = A_0/2 + A_1 \operatorname{Cos}(t) + A_2 \operatorname{Cos}(2t) + A_3 \operatorname{Cos}(3t) + \ldots$$
$$+ B_1 \operatorname{Sin}(t) + B_2 \operatorname{Sin}(2t) + B_3 \operatorname{Sin}(3t) + \ldots$$

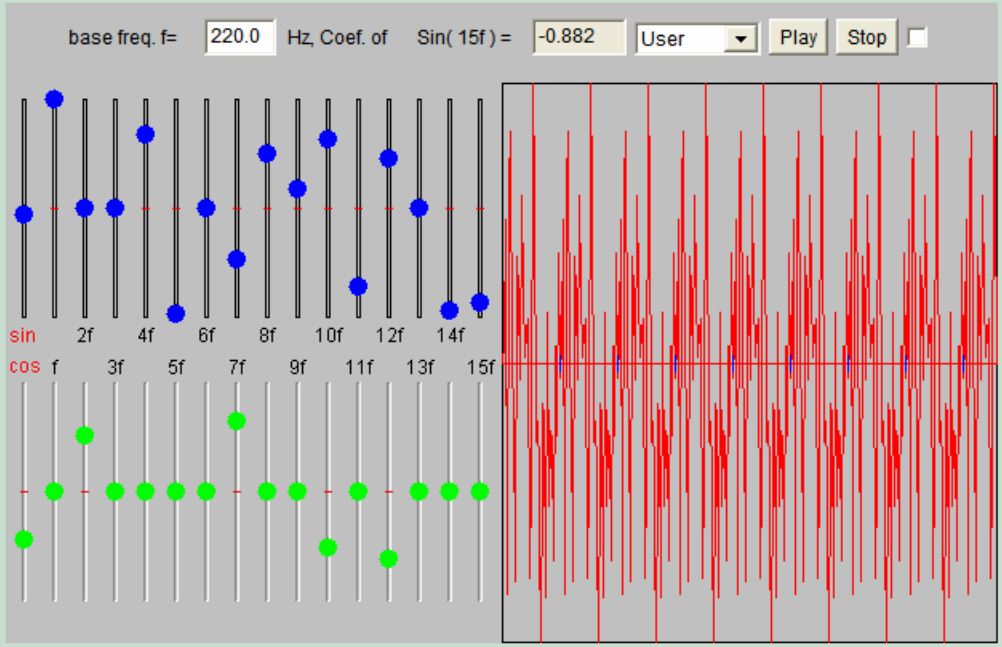
For coefficients that go to 0 fast enough these sums will converge at each value of t.

This defines a new function, which must be a ***periodic function***. (Period 2π)

Fourier's claim: *ANY* periodic function *f(t)* can be written this way

Music

http://www.phy.ntnu.edu.tw/ntnujava/viewtop



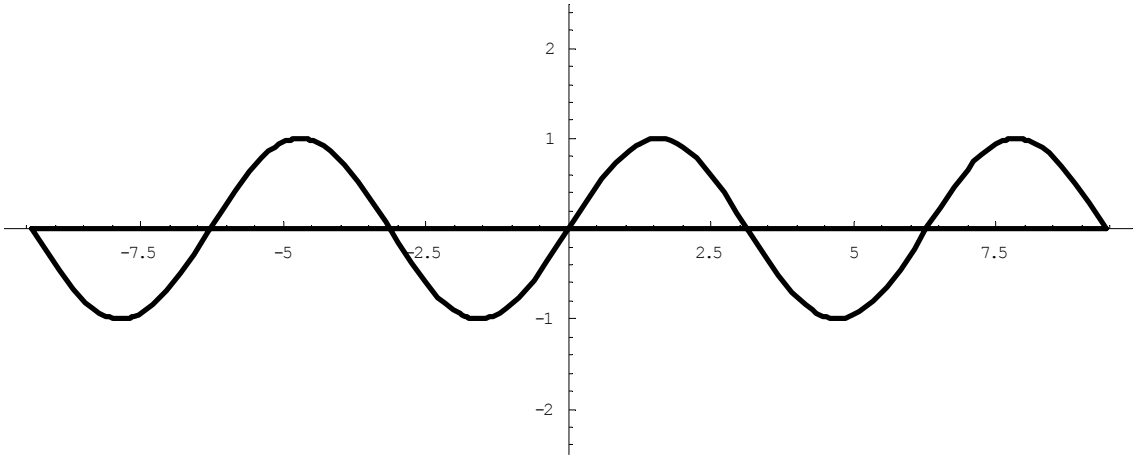
Click **get** button to fetch coefficients. Click **set** button to modify coefficients.

set	0f	f	2f	3f	4f	5f	6f	7f
sin	get	1.0	0.0	0.0	0.69	-0.98	0.0	-0.48
cos	-0.45	0.0	0.52	0.0	0.0	0.0	0.0	0.66
	8f	9f	10f	11f	12f	13f	14f	15f
sin	0.51	0.19	0.64	-0.73	0.47	0.0	-0.95	-0.88

Fourier's Representation

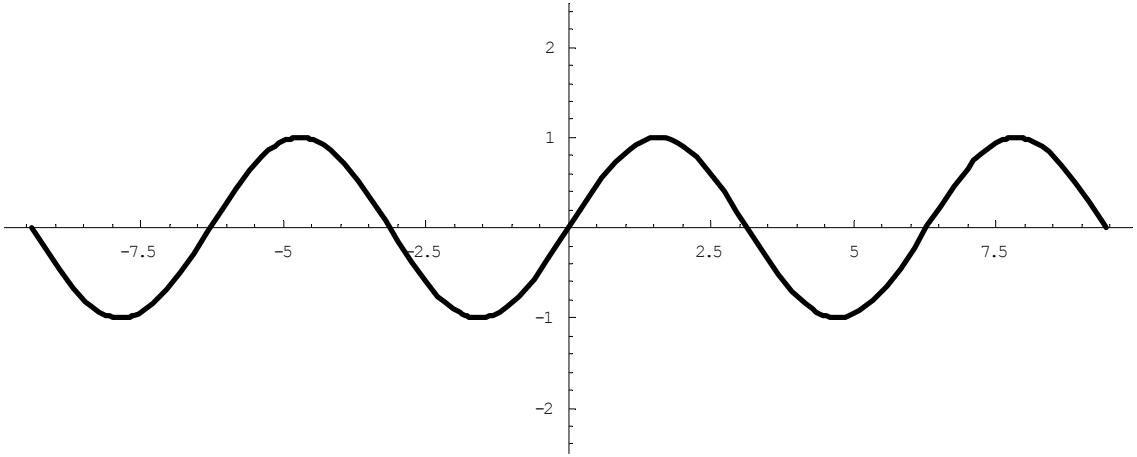
Represent $f(t) = t$ for $|t| < \pi$ and 2π periodic

$\mathrm{Sin}(t) + \ldots$



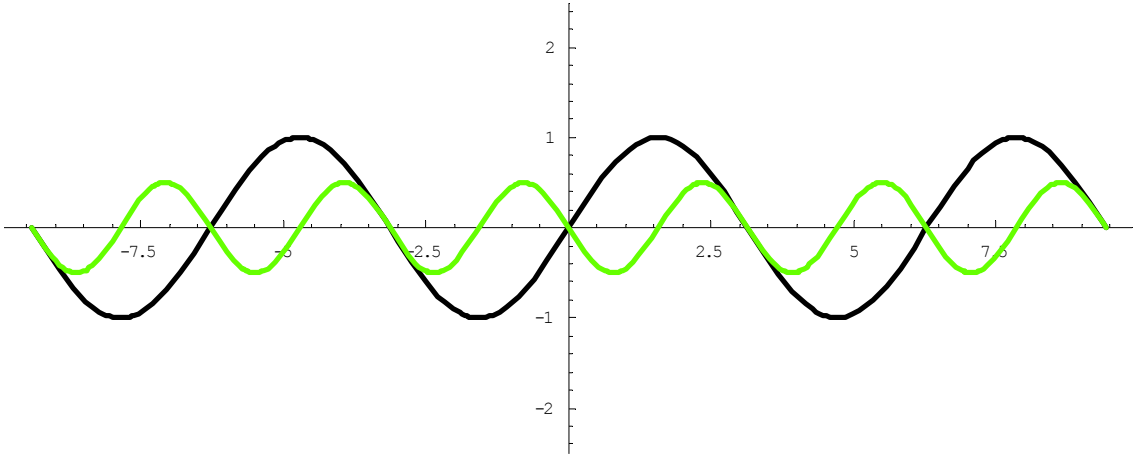
Fourier's Representation

1 Sin(t) + …



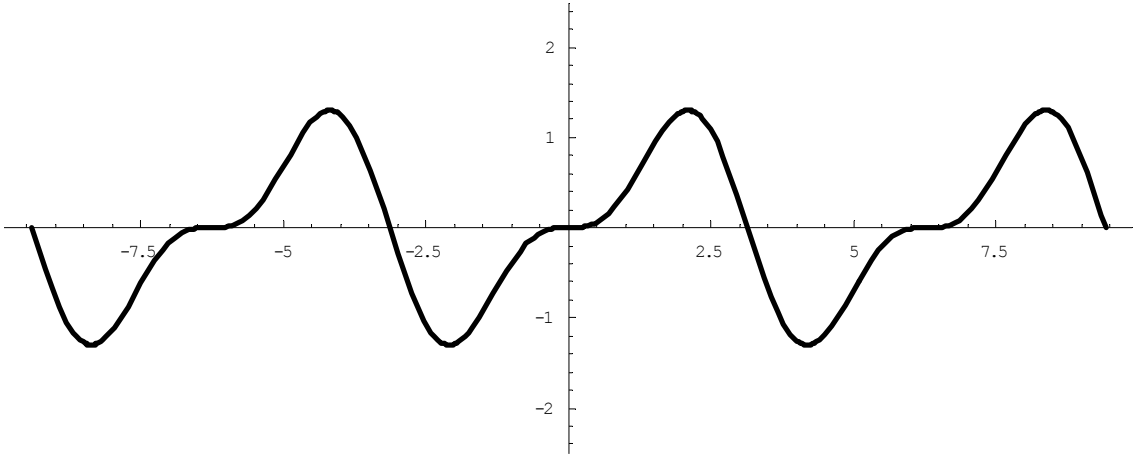
Fourier's Representation

Sin(t) - 1/2 Sin(2 t) + …



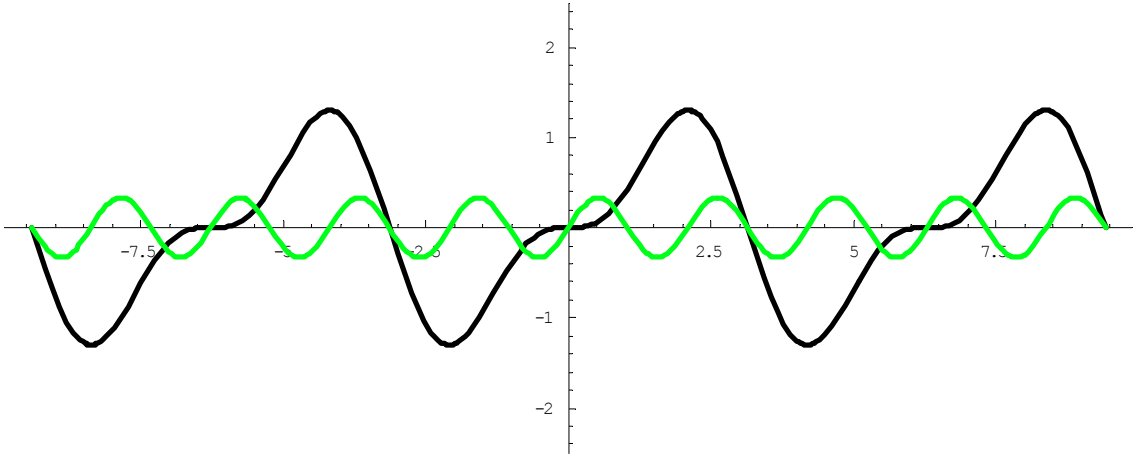
Fourier's Representation

1 Sin(t) - 1/2 Sin(2 t) + …



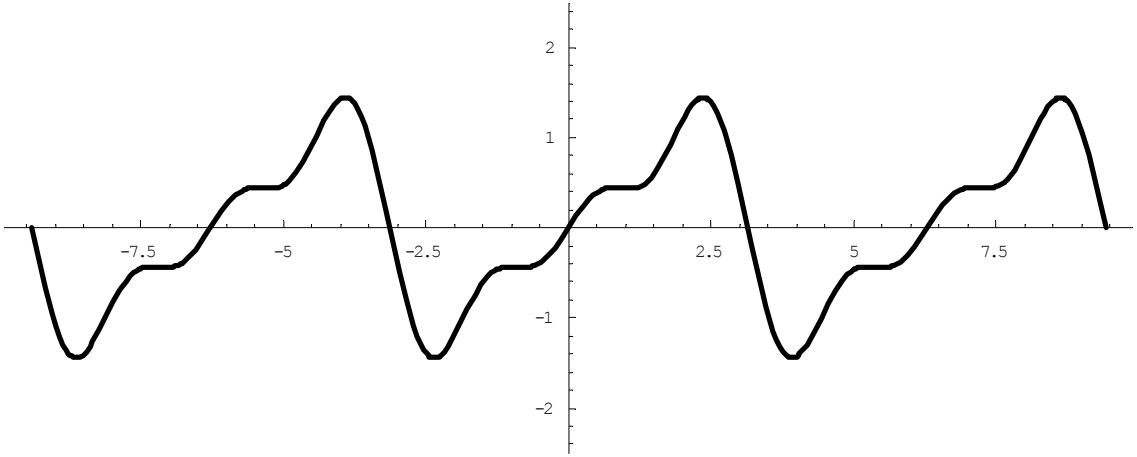
Fourier's Representation

$$1 \operatorname{Sin}(t) - 1/2 \operatorname{Sin}(2t) + 1/3 \operatorname{Sin}(3t) + \ldots$$



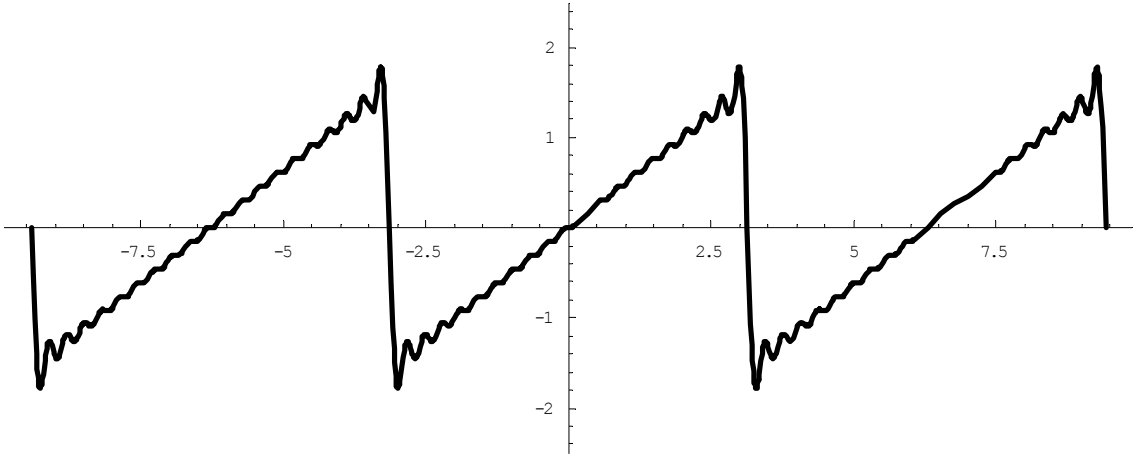
Fourier's Representation

$$1 \operatorname{Sin}(t) - 1/2 \operatorname{Sin}(2t) + 1/3 \operatorname{Sin}(3t) + \ldots$$



Fourier's Representation

20'th degree Fourier expansion



How do you get the Coefficients for a given f ?

$$A_0/2 + A_1 \operatorname{Cos}(t) + A_2 \operatorname{Cos}(2t) + A_3 \operatorname{Cos}(3t) + \ldots$$

$$+ B_1 \operatorname{Sin}(t) + B_2 \operatorname{Sin}(2t) + B_3 \operatorname{Sin}(3t) + \ldots$$

Fourier's claim:

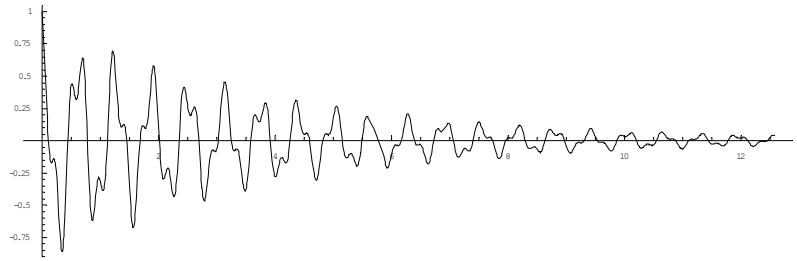
- *ANY* periodic function *f(t)* can be written this way (SYNTHESIS)
- The coefficients are uniquely determined by f: (ANALYSIS)

$$A_k = 1/2\pi \int_0^{2\pi} f(t) \cos(kt)\, dt$$

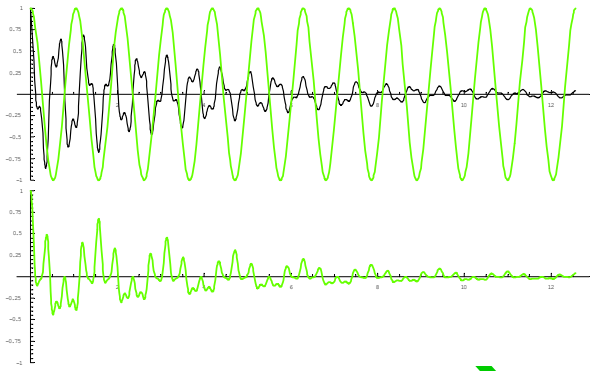
$$B_k = 1/2\pi \int_0^{2\pi} f(t) \sin(kt)\, dt$$

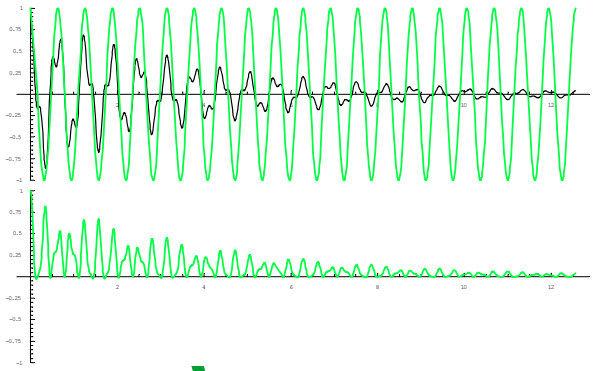
Fourier Analysis: match data with sinusoids

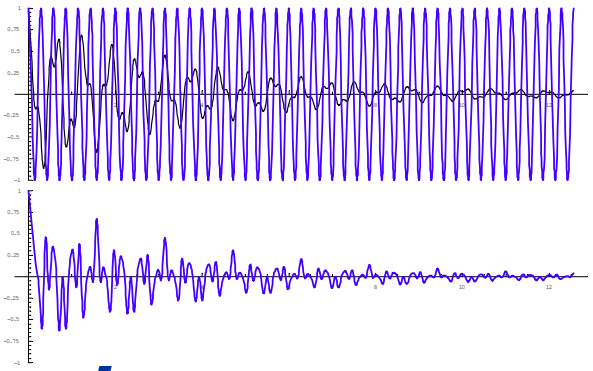
s(t)



Time t

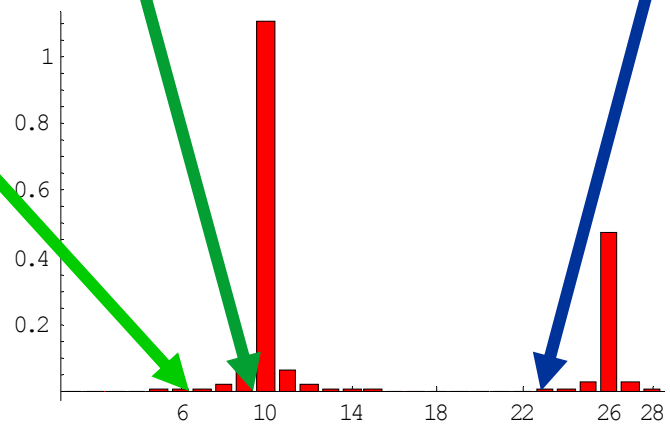






S[k]

$$= \int s(t) \cos(k\,t)\, d\,t$$



Frequency k

Complex Notation

For f, periodic with period p

Fourier transform f(t) -> F[k]

$$F[k] = 1/p \int_0^p f(t)\, e^{-2\pi i k t/p}\, dt$$

$$= 1/p \int_0^p f(t) \cos(2\pi k t/p)\, dt$$

$$- i/p \int_0^p f(t) \sin(2\pi k t/p)\, dt$$

Inverse Fourier transform F[k] -> f(t)

$$f(t) = \sum_{k \text{ in } Z} F[k]\, e^{2\pi i k t/p}$$

Sampling

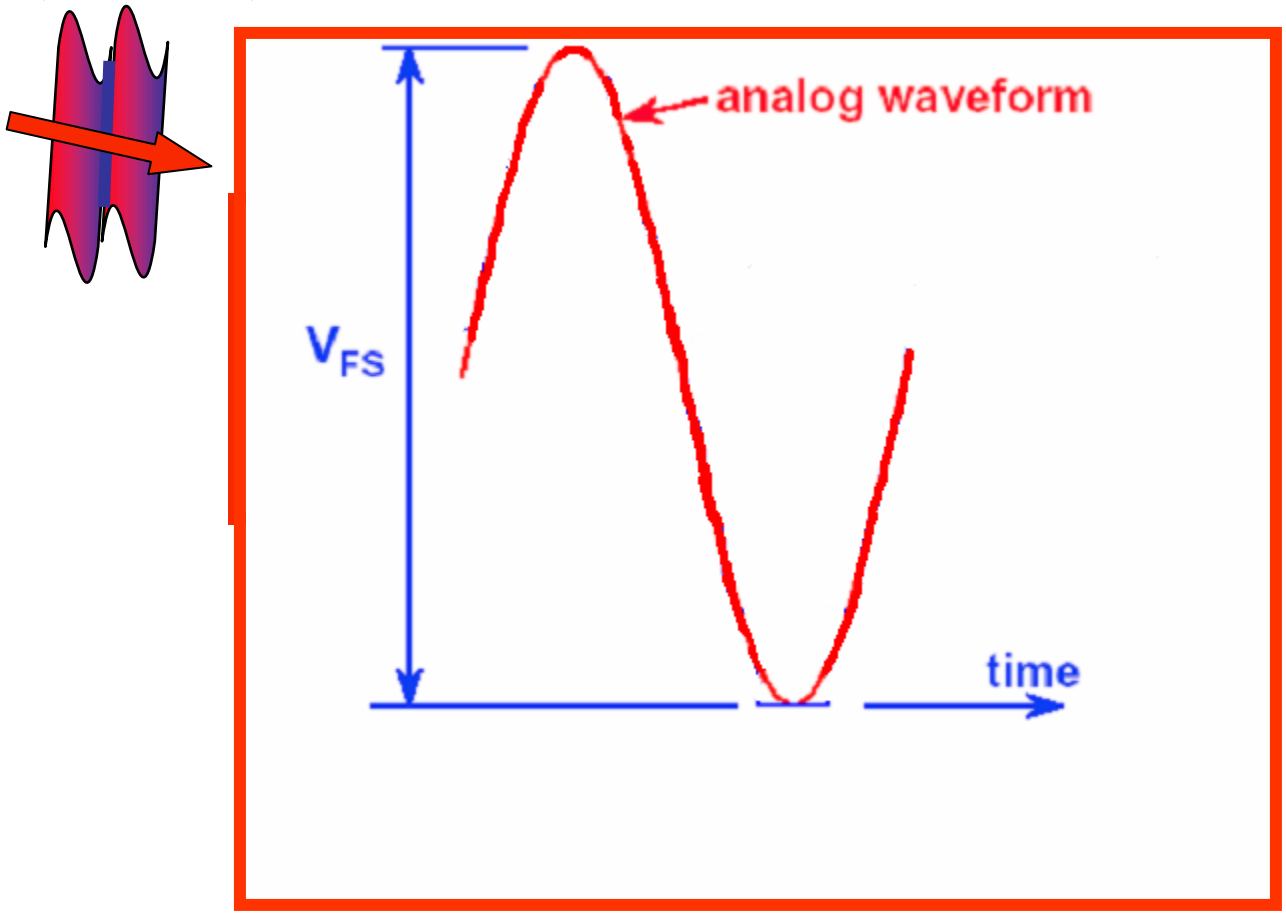
Fourier representations work just fine with sampled data

Simple connection to Fourier of the continuous function it came form

Familiar example: Digital Audio

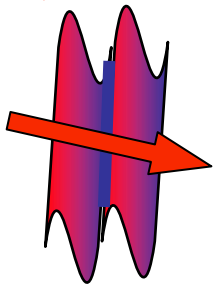
Measuring and Discretizing Input field

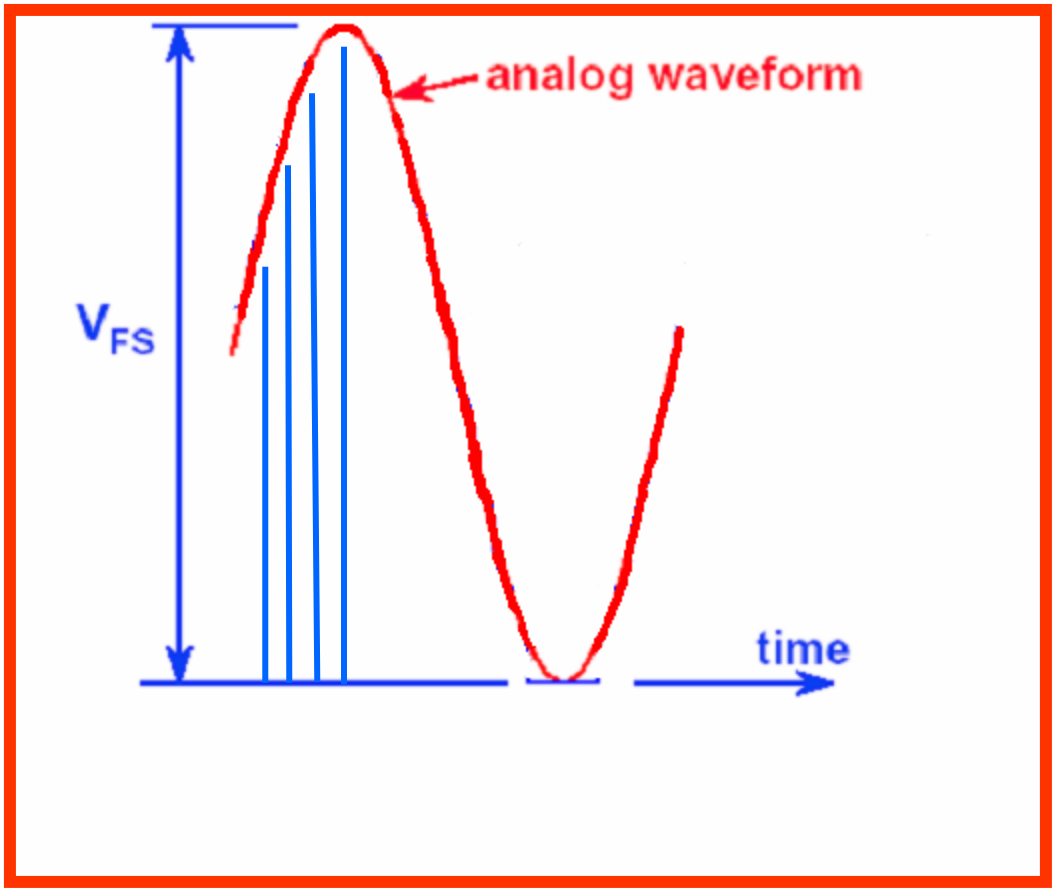
Physical Field
(continuum)



Sample

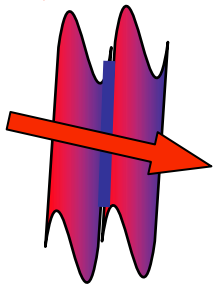
Physical Field
(continuum)



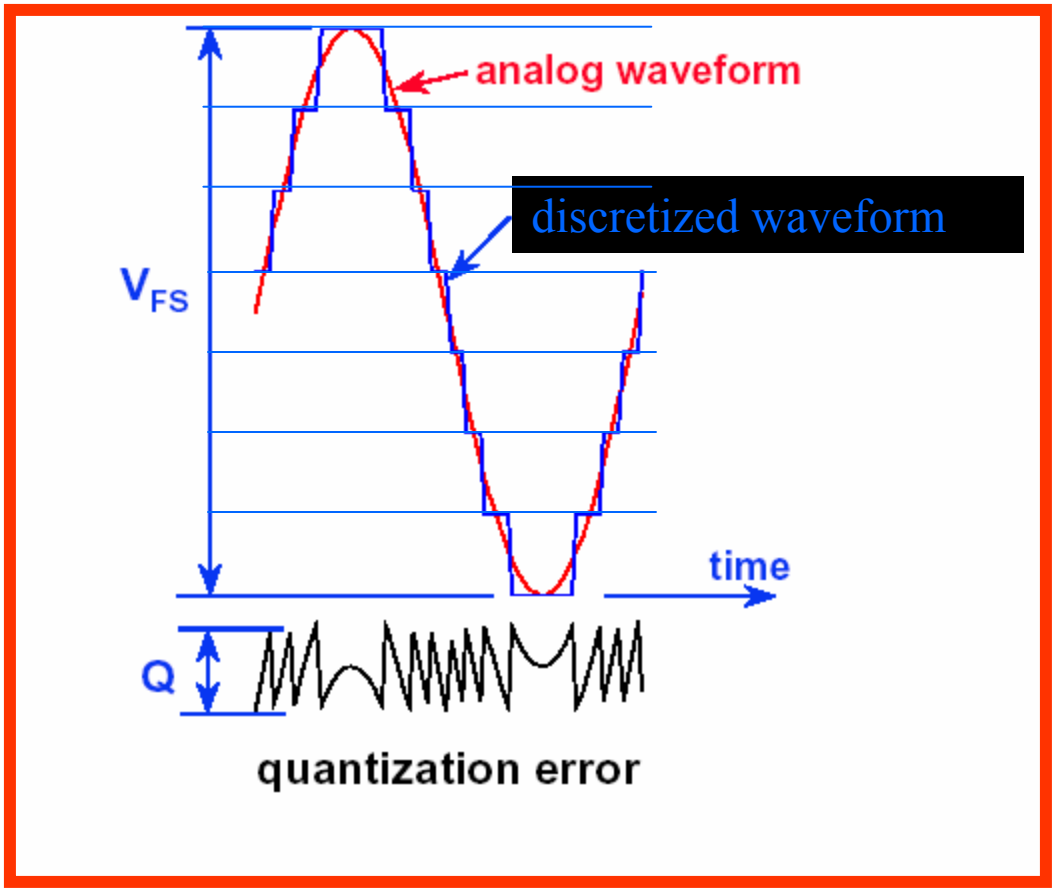


Quantize

Physical Field
(continuum)

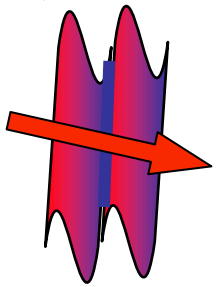


$$NP_0(rms) = \frac{Q}{\sqrt{12}}$$



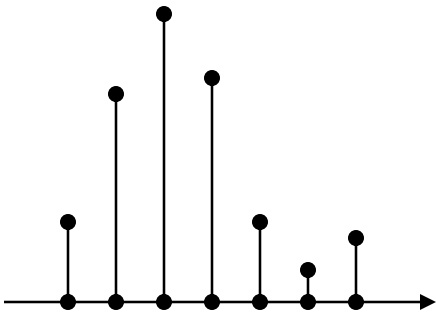
Code and output

Physical Field
(continuum)

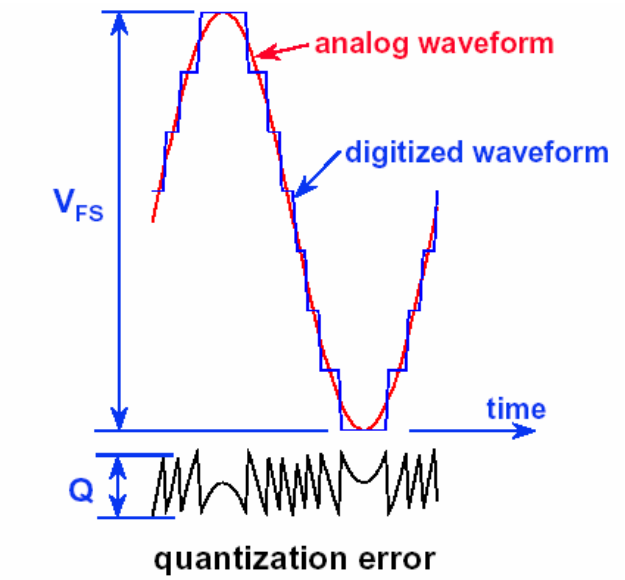


PHYSICAL LAYER

Digital Representation

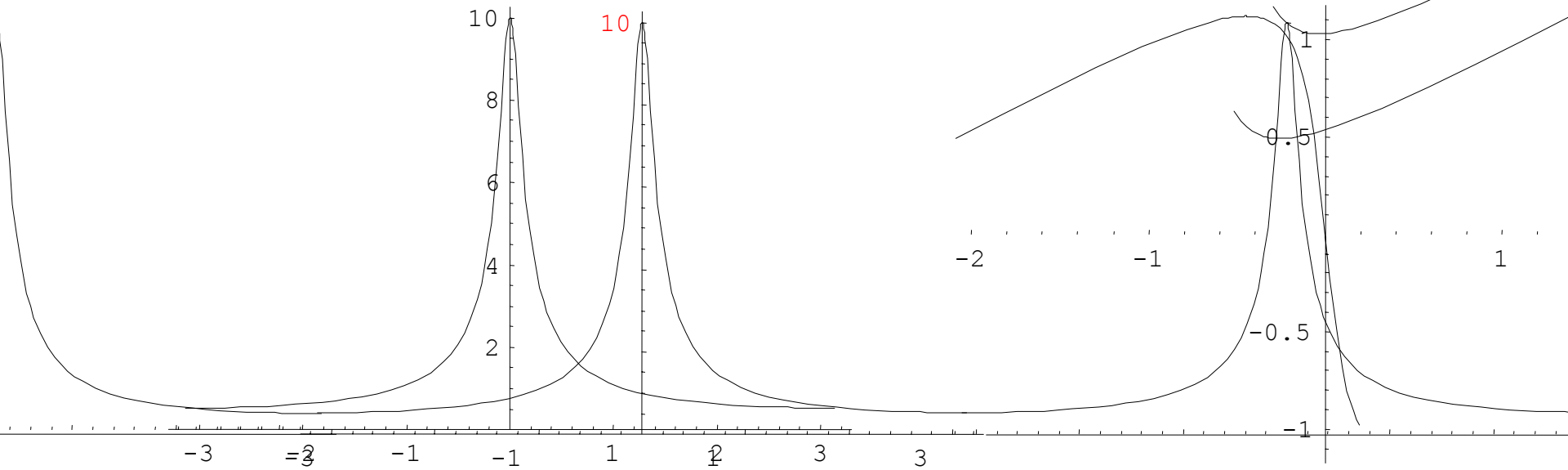


…3, 8, 10, 9, 3, 1, 2…



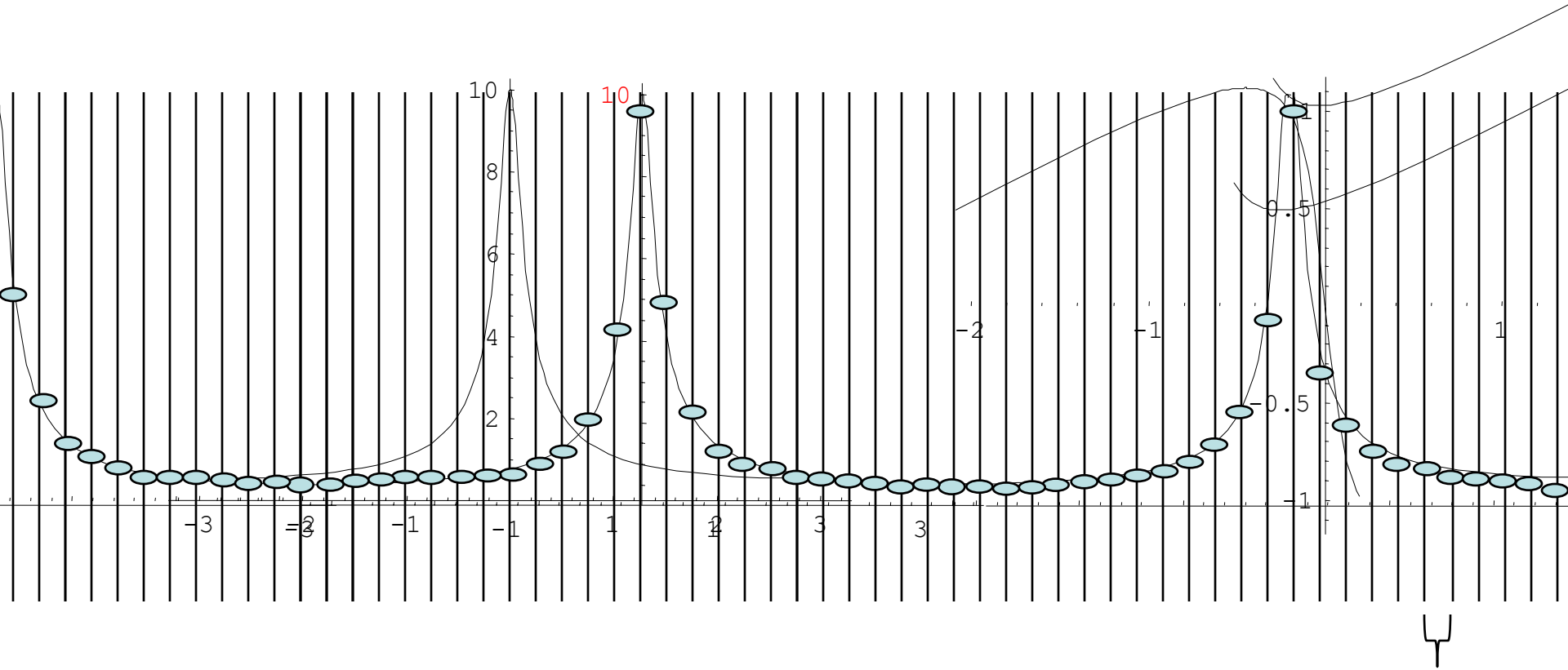
Sampling

Often must work with a discrete set of measurements of a continuous function



Sampling

Takes a function defined on $\mathbb{R}$ and creates a function defined on $\mathbb{Z}$

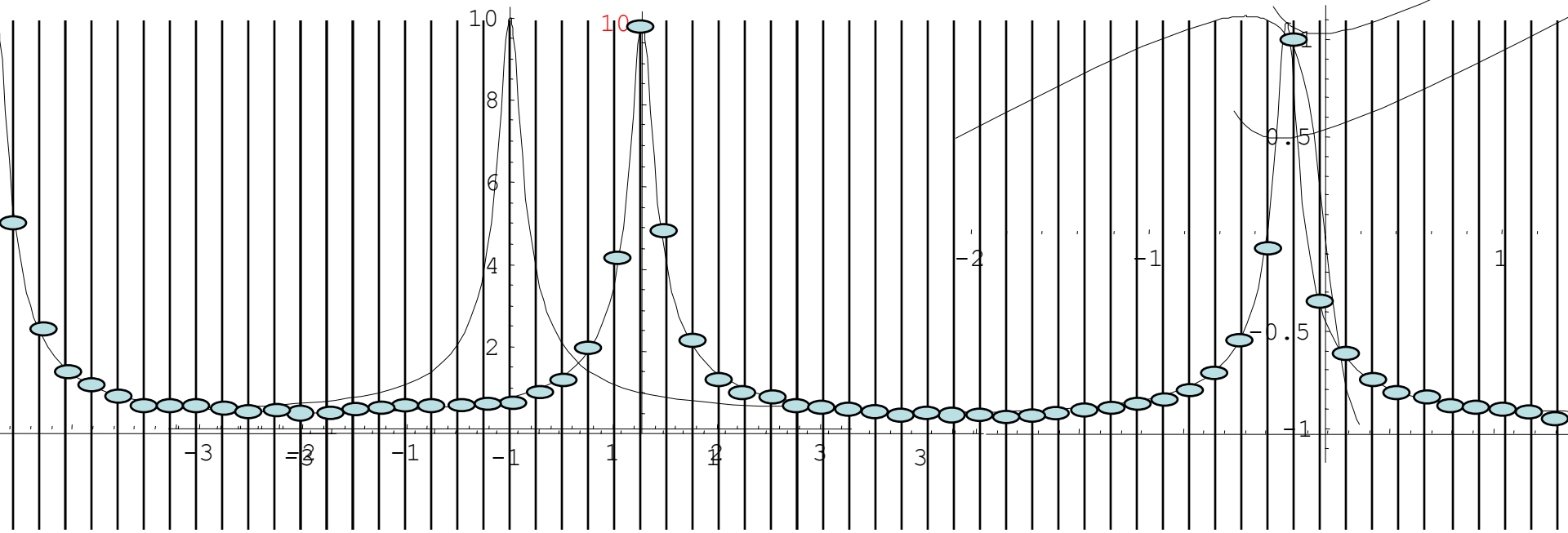


$\mathbf{S}_h$ $\quad f(t) \longrightarrow \phi[n] = f(n\,h)$

Sampling

In this case, it is a periodic function on Z,

(Assuming p/h = N)

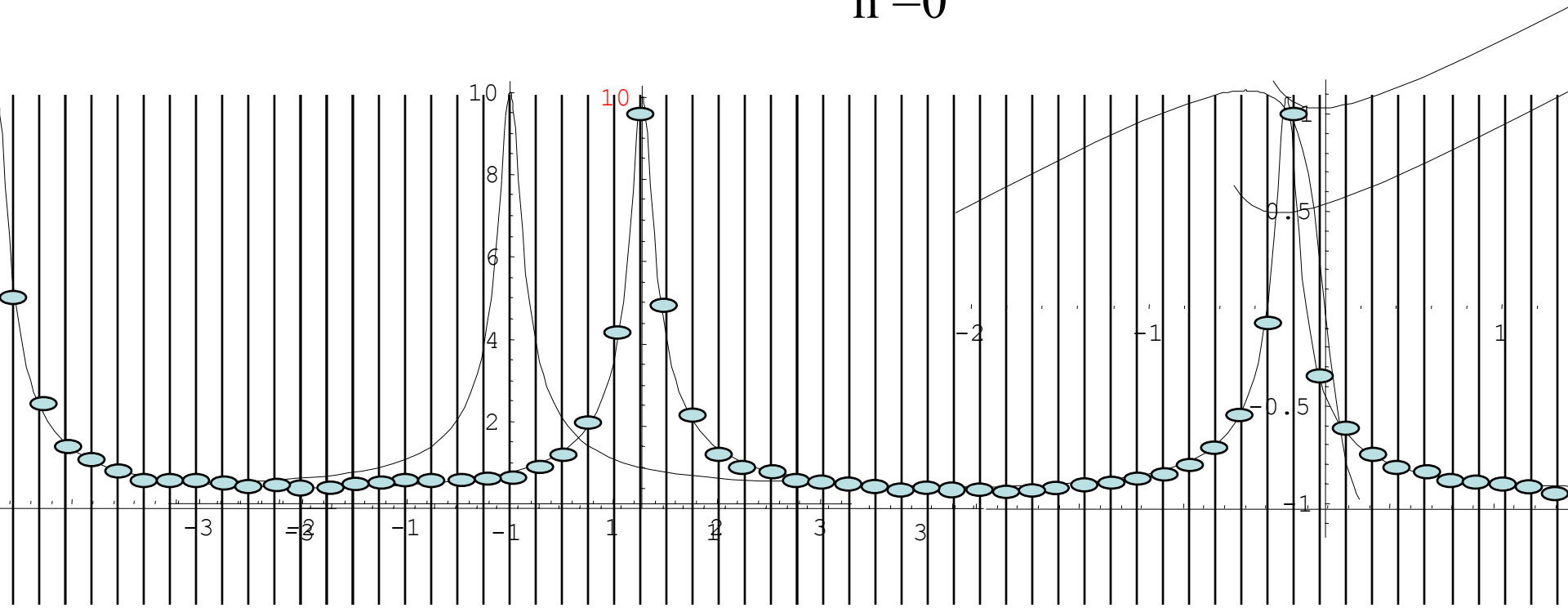


0 1 2 3 ….. N

$\phi[n] = f(n\,h)$ $\phi[n+N] = \phi[n]$ h

DFT

$$\int_0^p f(t)\, e^{-2\pi i k t/p}\, dt \longrightarrow \sum_{n=0}^{N-1} \phi[n]\, e^{-2\pi i k n h/p}$$



$$f(t) \longrightarrow \phi[n] = f(n\,h)$$

h

DFT and its inverse for periodic discrete data

$$\Phi[k] = \sum_{n=0}^{N-1} \phi[n]\, e^{-2\pi i k n h/p} \qquad p = N h$$

$$= \sum_{n=0}^{N-1} \phi[n]\, e^{-2\pi i k n/N}$$

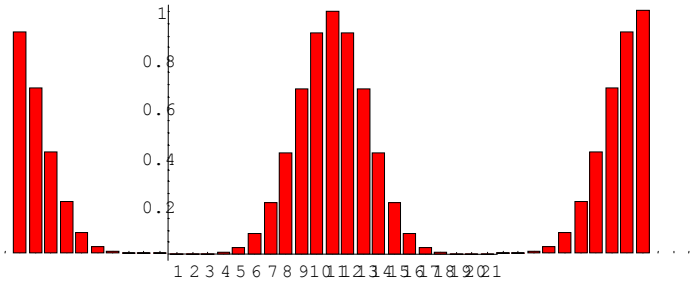
This is automatically periodic in k with period N

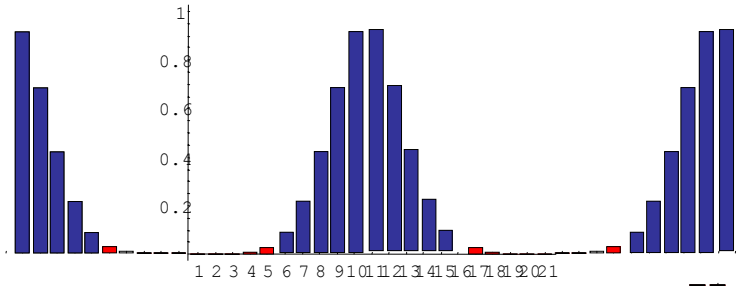
Inverse is like Fourier series, but with only p terms

DFT: Discrete time periodic version of Fourier

"time" domain *"frequency" domain*

$$1/N \sum_{k=0}^{N} \gamma[k]\, e^{-2\pi i k m/N} = \Gamma[m]$$





$$\gamma[k] = \sum_{k=0}^{N} \Gamma[k]\, e^{2\pi i m k/N}$$

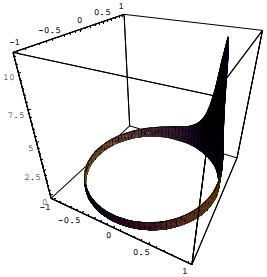
$\gamma[k]$, on P_N

i.e. on Z, Period N

$\Gamma[m]$ on P_N

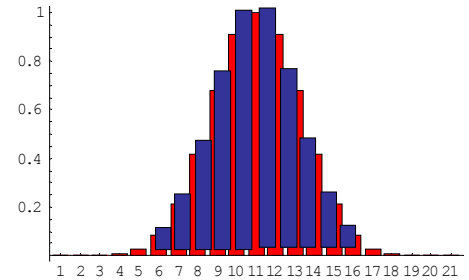
i.e. on Z, Period N

Fourier



$$\int_0^p f(t)\, e^{-2\pi i k t/p}\, dt/p$$

$$\sum_{k \text{ in } \mathbb{Z}} F[k]\, e^{2\pi i k t/p}$$



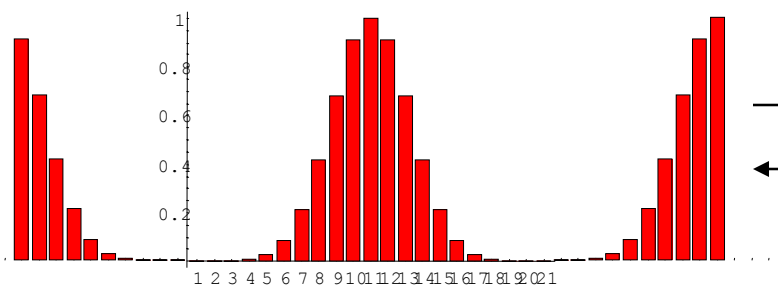
f(t), period p

F[k] , on $\mathbb{Z}$

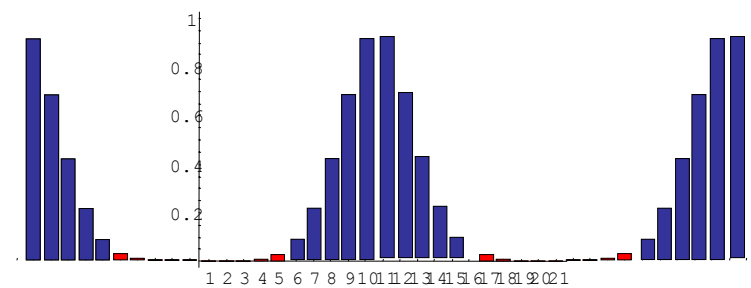
$S_{p/N}$

P_N

$$1/N \sum_{k=0}^{N-1} \gamma[k]\, e^{-2\pi i k m/N}$$



k =0



$\gamma[k]$, on P_N

$$\sum_{k=0}^{N-1} \Gamma[k]\, e^{2\pi i m k/N}$$

Γ [m] on P_N

i.e. on $\mathbb{Z}$, Period N

i.e. on $\mathbb{Z}$, Period N

"time" domain

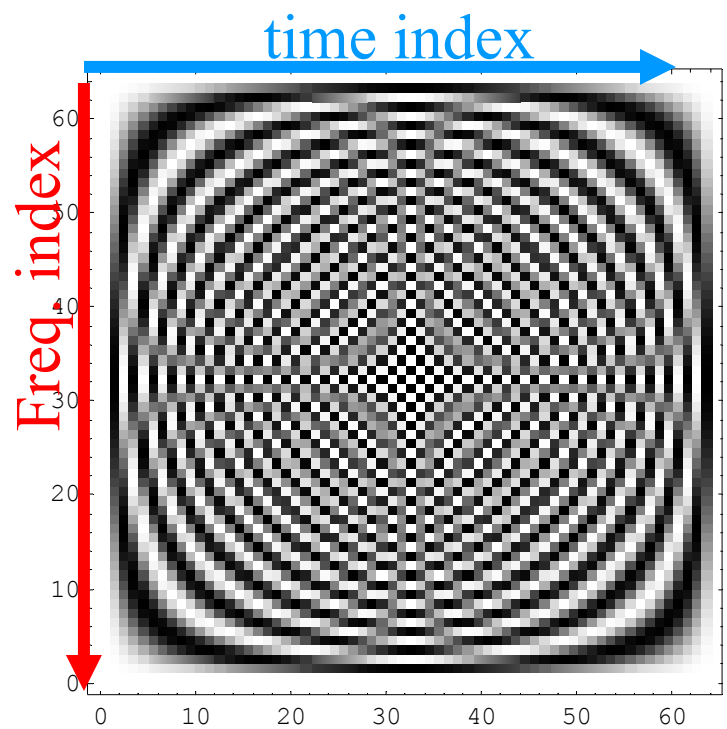
"frequency" domain

Discrete time Numerical Fourier Analysis

DFT is really just a matrix multiplication!

$$\Gamma[m] = 1/N \sum_{k=0}^{N-1} e^{-2\pi i k m/N} \gamma[k]$$

$$\begin{pmatrix} \Gamma[0] \\ \Gamma[1] \\ \Gamma[2] \\ \vdots \\ \Gamma[N-1] \end{pmatrix} =$$



$$\begin{pmatrix} \gamma[0] \\ \gamma[1] \\ \gamma[2] \\ \vdots \\ \gamma[N-1] \end{pmatrix}$$

$$\Gamma = F_N \gamma$$

Numerical Harmonic Analysis

FFT: Symmetry Properties permits "Divide and Conquer"
Sparse Factorization

$$F_{mn} = (F_m \otimes I_n) \cdot T_n^{mn} \cdot (I_m \otimes F_n) \cdot L_m^{mn}$$