Introduction to Physical Acoustics

Class webpage

- CMSC 828D: Algorithms and systems for capture and playback of spatial audio.

www.umiacs.umd.edu/~ramani/cmsc828d_audio

- Send me a test email message with the subject cmsc828d

Goals

- Physical Acoustics is the branch of physics studying propagation of sound
- Our goals: understand some background material about sound propagation

Fluid Mechanics 101

- Properties of Matter
  - Density ρ
  - Pressure p
  - Compressibility ($dp/d\rho$)
  - viscosity
- Conservation Laws
  - Mass is conserved (in the absence of sources)
  - Momentum is conserved (F=Ma)
  - Energy is conserved
- Three Conservation Laws describe how imposed changes affect a fluid
- Treat the fluid as a continuum subject to the equations of continuum mechanics
- Equations governing acoustics will be a special (simpler) case of these equations

Mathematical Modeling

- One of the extraordinary successes of the 19^{th} and 20^{th} centuries is the development of mathematical models to predict the behavior of fluid and solid media
- Aircraft, automobile, buildings, mechanical design of all products, engines etc. based on this understanding

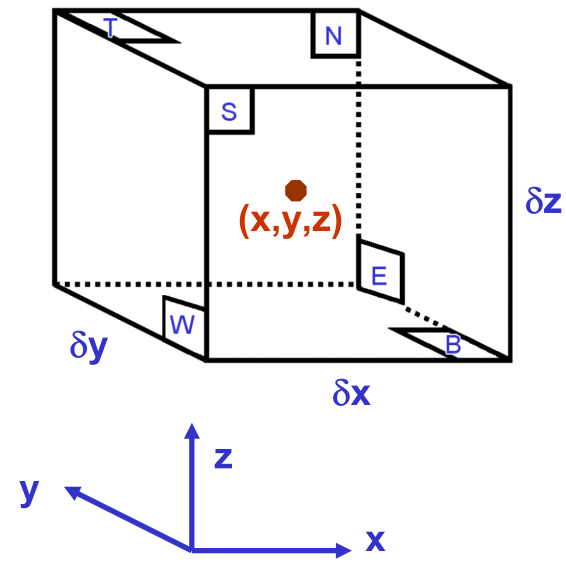
Conservation of Mass

- Derivation
- Consider a box of size $\delta x \times \delta y \times \delta z$ through which fluid flows
- It has a density ρ ($\mathbf{x}$) and the flow vector $\mathbf{u}$=(u,v,w)

Fluid element and properties

- The behavior of the fluid is described in terms of macroscopic properties:
  - Velocity **u.**
  - Pressure p.
  - Density ρ.
  - Temperature T.
  - Energy E.
- Typically ignore (x,y,z,t) in the notation.
- Properties are averages of a sufficiently large number of molecules.
- A fluid element can be thought of as the smallest volume for which the continuum assumption is valid.

Fluid element for conservation laws

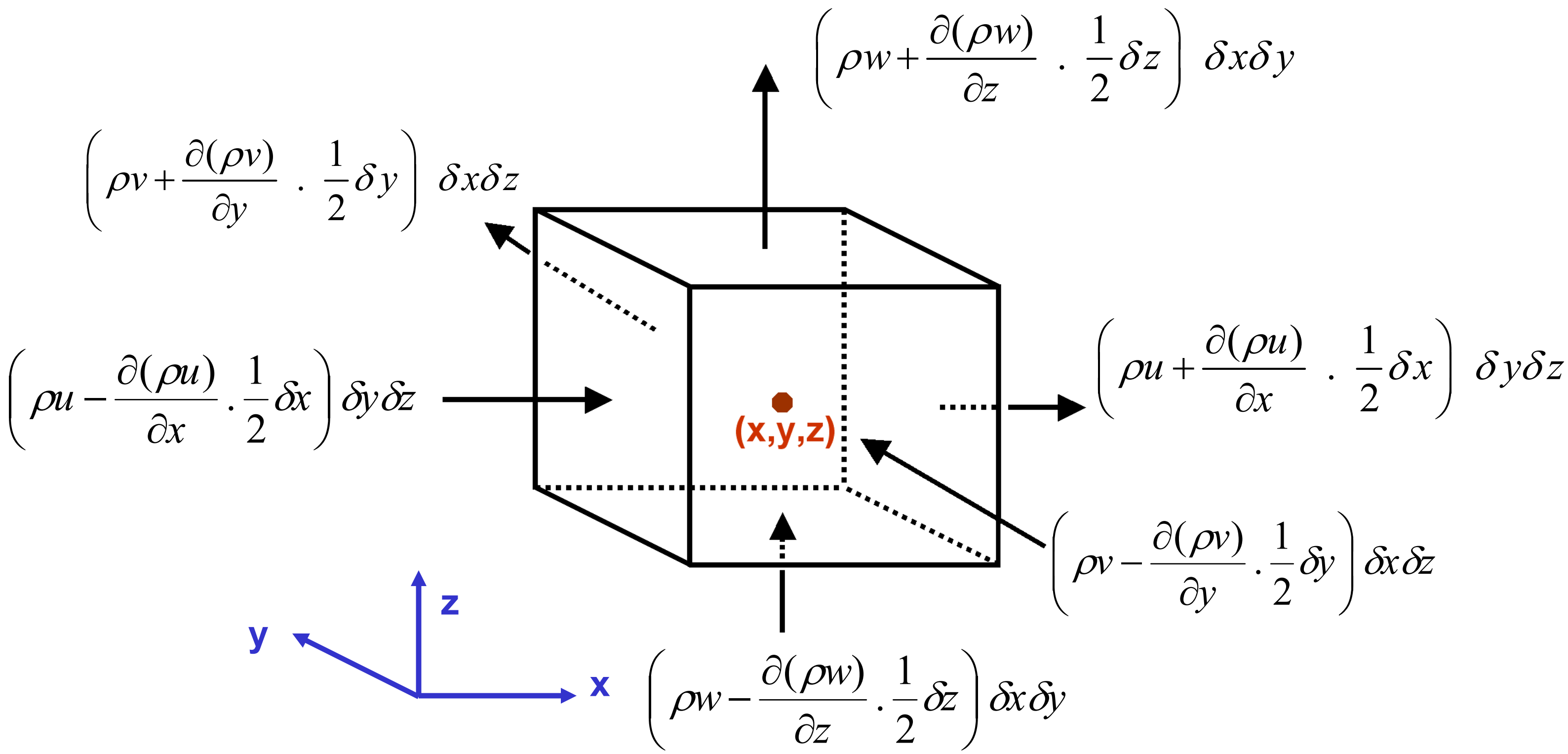


Faces are labeled North, East, West, South, Top and Bottom

Properties at faces are expressed as first two terms of a Taylor series expansion, e.g. for p: $p_W = p - \frac{\partial p}{\partial x}\frac{1}{2}\delta x$ and $p_E = p + \frac{\partial p}{\partial x}\frac{1}{2}\delta x$

Mass balance

- Rate of increase of mass in fluid element equals the net rate of flow of mass into element.
- Rate of increase is: $\frac{\partial}{\partial t}(\rho \delta x \delta y \delta z) = \frac{\partial \rho}{\partial t} \delta x \delta y \delta z$
- The inflows (positive) and outflows (negative) are shown here:



Mass Conservation ("Continuity") equation

- Summing all terms in the previous slide and dividing by the volume $\delta x \delta y \delta z$ results in:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

- In vector notation: $\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$

Change in density

Convective term: flow of mass out

Creation of mass

- For incompressible constant property fluids $\partial\rho/\partial t = 0$, and $\nabla \rho$ = 0 the equation becomes: $div\ \mathbf{u} = 0$.
- Alternative ways to write this: $\frac{\partial u_i}{\partial x_i} = 0$

Rate of change for a stationary fluid element

- In most cases we are interested in the changes of a flow property for a fluid element, or fluid volume, that is stationary in space.
- However, some equations are easier derived for fluid particles. For a moving fluid particle, the total derivative per unit volume of this property ϕ is given by:

(for moving fluid particle) $$\rho \frac{D\phi}{Dt} = \rho \left(\frac{\partial \phi}{\partial t} + \mathbf{u}.grad\ \phi \right)$$ **(for given location in space)**

- For a fluid element, for an arbitrary conserved property ϕ:

$$\frac{\partial \rho}{\partial t} + div(\rho\, \mathbf{u}) = 0$$

Continuity equation

$$\frac{\partial (\rho \phi)}{\partial t} + div\, (\rho \phi\, \mathbf{u}) = 0$$

Arbitrary property

Relevant entries for *Φ*

x-momentum	u	$\rho \frac{Du}{Dt}$	$\frac{\partial(\rho u)}{\partial t} + div(\rho u \mathbf{u})$
y-momentum	v	$\rho \frac{Dv}{Dt}$	$\frac{\partial(\rho v)}{\partial t} + div(\rho v \mathbf{u})$
z-momentum	w	$\rho \frac{Dw}{Dt}$	$\frac{\partial(\rho w)}{\partial t} + div(\rho w \mathbf{u})$
Energy	E	$\rho \frac{DE}{Dt}$	$\frac{\partial(\rho E)}{\partial t} + div(\rho E \mathbf{u})$

Conservation Laws

- Equations in a gas (like air)
- Variables
  - ρ – density
  - p – pressure
  - **u** – velocity
  - T – temperature
- Constants
  - C_p is "heat capacity"

Mass

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0,$$

Momentum for an inviscid °uid

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla(\mathbf{u}) + \nabla p = 0$$

Energy (neglecting heat conduction)

$$\rho C_p \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla(\rho T) \right) - \frac{Dp}{Dt} = 0.$$

Equation of state relates three of the quantities

$$\rho = \frac{p}{RT}$$

Conservation Laws

- Eliminate ρ
- Introduce the short hand notation
-

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\left(\frac{p}{RT}\right)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

- Yields the system of equations

$$\frac{1}{p}\frac{Dp}{Dt} - \frac{1}{T}\frac{DT}{Dt} + \nabla \cdot \mathbf{u} = 0$$

$$\frac{p}{RT}\frac{D\mathbf{u}}{Dt} + \nabla p = 0$$

$$\frac{1}{T}\frac{DT}{Dt} - \frac{\gamma - 1}{\gamma p}\frac{Dp}{Dt} = 0$$

- γ is the ratio of specific heats for the gas

Equations of Acoustics

- Acoustics govern the propagation of small perturbations through the system
- Let the system be in equilibrium with pressure p_0, Temperature T_0, and zero velocity
- Then a small disturbance p' will upset the equilibrium
- Using conservation laws, we can derive the equations of acoustics that govern the propagation of sound waves in the medium.
- Key assumption ("acoustic approximation")
- All perturbations are much smaller than equilibrium values

Acoustic equations

- Equations under these assumptions are\

$$\frac{1}{p_0}\frac{\partial p}{\partial t} - \frac{1}{T_0}\frac{\partial T}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

$$\frac{p_0}{RT_0}\frac{\partial \mathbf{u}}{\partial t} + \nabla p = 0$$

$$\frac{1}{T_0}\frac{\partial T}{\partial t} - \frac{\gamma - 1}{\gamma p_0}\frac{\partial p}{\partial t} = 0$$

- Can eliminate T from the system

$$\frac{1}{\gamma p_0}\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

$$\frac{p_0}{RT_0}\frac{\partial \mathbf{u}}{\partial t} + \nabla p = 0$$

The wave equation

- Almost there …
- Differentiate first equation with respect to time

$$\frac{1}{\gamma p_0}\frac{\partial^2 p}{\partial t^2} + \nabla \cdot \frac{\partial \mathbf{u}}{\partial t} = 0$$

- Substitute for ∂ **u** /∂ t from second equation

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p$$

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla \cdot \nabla p = 0$$

$c^2 = \frac{\gamma p_0}{\rho_0}$ Here c is the sound speed

Laplace operator

- $\nabla \cdot \nabla = \nabla^2$ is the Laplace operator
- (Divergence of gradient)
- Extremely common in partial differential equations
- So the wave equation can be written as

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0$$

Wave equation for the Velocity potential

- Let ϕ be the velocity potential so that **u**=∇ ϕ
- Then the conservation equations become
- From these we can eliminate p to arrive at the wave equation for the velocity potential

$$\frac{1}{\gamma p_0}\frac{\partial p}{\partial t} + \nabla^2 \phi = 0$$

$$\nabla \left(\frac{p_0}{RT_0}\frac{\partial \phi}{\partial t} + p \right) = 0$$

$$-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \nabla^2 \phi = 0$$