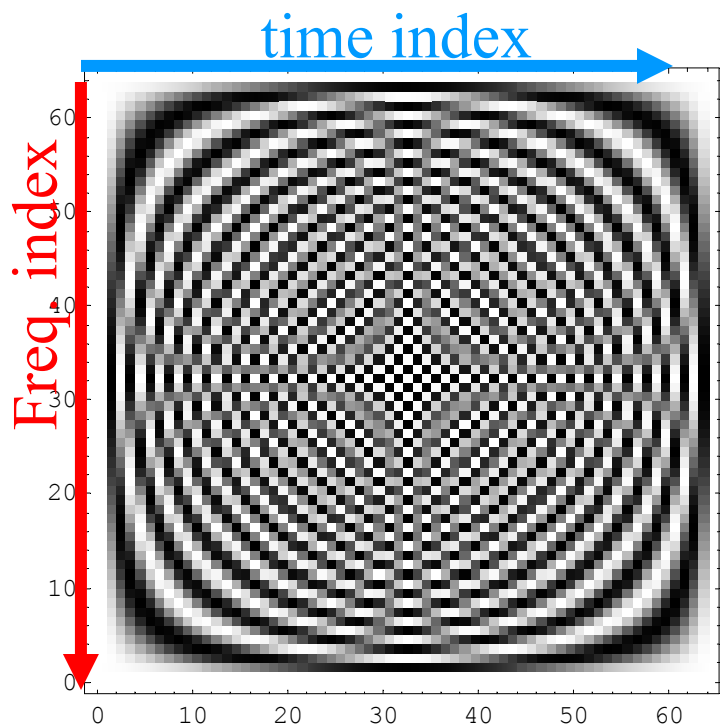
Discrete time Numerical Fourier Analysis

DFT is really just a matrix multiplication!

$$\Gamma[m] = 1/N \sum_{k=0}^{N-1} e^{-2\pi i k m/N} \gamma[k]$$

$$\begin{pmatrix} \Gamma[0] \\ \Gamma[1] \\ \Gamma[2] \\ \vdots \\ \Gamma[N-1] \end{pmatrix} =$$



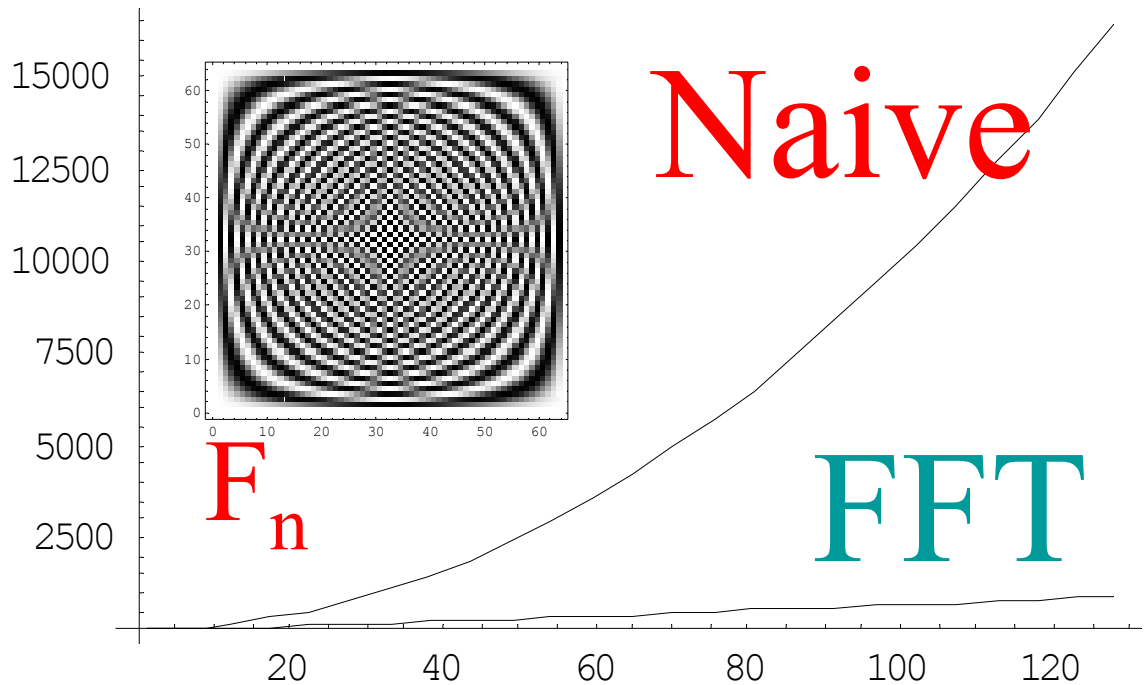
$$\begin{pmatrix} \gamma[0] \\ \gamma[1] \\ \gamma[2] \\ \vdots \\ \gamma[N-1] \end{pmatrix}$$

$$\Gamma = F_N \gamma$$

Numerical Harmonic Analysis

FFT: Symmetry Properties permits "Divide and Conquer"
Sparse Factorization

$$F_{mn} = (F_m \otimes I_n) \cdot T_n^{mn} \cdot (I_m \otimes F_n) \cdot L_m^{mn}$$



Structured matrices

- Fast algorithms have been found for many dense matrices
- Typically the matrices have some *"structure"*
- Definition:
  - A dense matrix of order $N \times N$ is called structured if its entries depend on only $O(N)$ parameters.
- Most famous example – the fast Fourier transform

Fourier Matrices

A Fourier matrix of order n is defined as the following

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{bmatrix},$$

where

$$\omega_n = e^{-\frac{2\pi i}{n}},$$

is an nth root of unity.

Fast Fourier Transform

- Fourier transform of a function *h(t)* is given by *H(f)* where *f* is the frequency

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t} df$$

- Discrete Fourier Transform: if the function is sampled at discrete times

$$h_k \equiv h(t_k), \qquad t_k \equiv k\Delta, \qquad k = 0, 1, 2, \ldots, N-1$$

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t} dt \approx \sum_{k=0}^{N-1} h_k \; e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k \; e^{2\pi i k n / N}$$

Discrete Fourier Transform

- All notion of time has disappeared
- Multiplication of sampled data by a matrix

$$H_n \equiv \sum_{k=0}^{N-1} h_k \, e^{2\pi i k n / N}$$

- This matrix is called the Fourier Matrix
- As discussed earlier it is a structured matrix

A Fourier matrix of order n is defined as the following

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{n-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & W^{n-1} & W^{2(n-1)} & \cdots & W^{(n-1)(n-1)} \end{bmatrix},$$

where

$$W = e^{\frac{2\pi i}{n}},$$

is an nth root of unity.

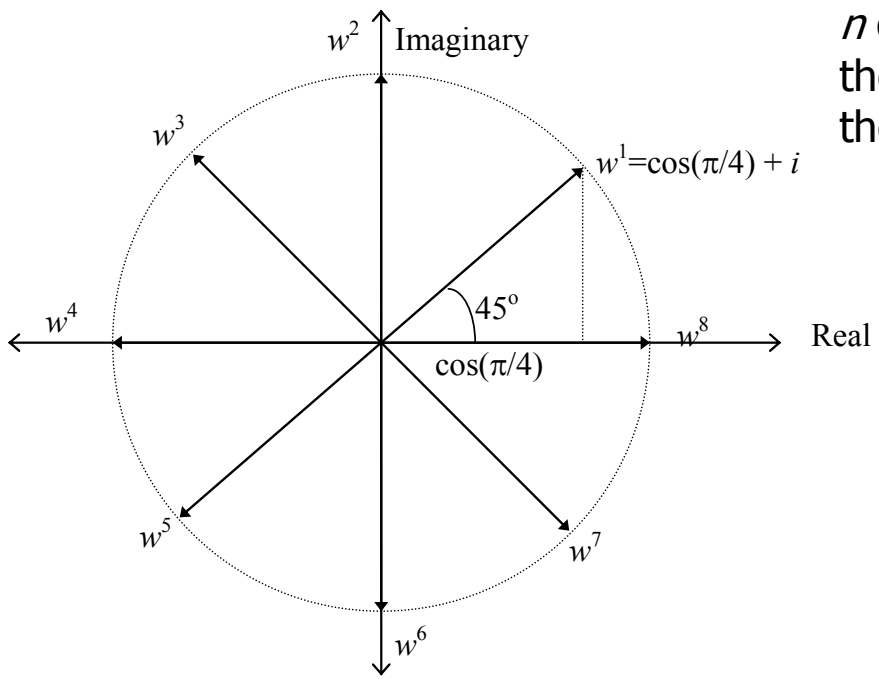
$i = \sqrt{-1}$ Primitive Roots of Unity

A number ω is a ***primitive n-th root of unity***, for n>1, if

$\omega^n = 1$

The numbers 1, ω, ω^2, …, ω^{n-1} are all distinct

- Example: The complex number $e^{2pi/n}$ is a primitive n-th root of unity, where



n complex roots of unity equally spaced around the circle of unit radius centered at the origin of the complex plane.

Roots of Unity: Properties

- Property 1: Let ω be the principal n^{th} root of unity. If $n > 0$, then $\omega^{n/2} = -1$.
  - Proof: $\omega = e^{2\pi i / n} \Rightarrow \omega^{n/2} = e^{\pi i} = -1$. (Euler's formula)
  - **Reflective Property:**
  - Corollary: $\omega^{k+n/2} = -\omega^{k}$.
- Property 2: Let $n > 0$ be even, and let ω and ν be the principal n^{th} and $(n/2)^{th}$ roots of unity. Then $(\omega^{k})^{2} = \nu^{k}$.
  - Proof: $(\omega^{k})^{2} = e^{(2k)2\pi i / n} = e^{(k)\, 2\pi i / (n/2)} = \nu^{k}$.
  - **Reduction Property:** If ω is a primitive (2n)-th root of unity, then ω^{2} is a primitive n-th root of unity.

L3: Let $n > 0$ be even. Then, the squares of the n complex n^{th} roots of unity are the n/2 complex $(n/2)^{th}$ roots of unity.

– Proof: If we square all of the nth roots of unity, then each $(n/2)^{th}$ root is obtained exactly twice since:

- L1 $\Rightarrow$ $\omega^{k+n/2} = -\omega^k$
- thus, $(\omega^{k+n/2})^2 = (\omega^k)^2$
- L2 $\Rightarrow$ both of these = ν^k
- $\omega^{k+n/2}$ and ω^k have the same square

- **Inverse Property:** If ω is a primitive root of unity, then $\omega^{-1}=\omega^{n-1}$
  – Proof: $\omega\omega^{n-1}=\omega^n=1$

Primitive Roots of Unity

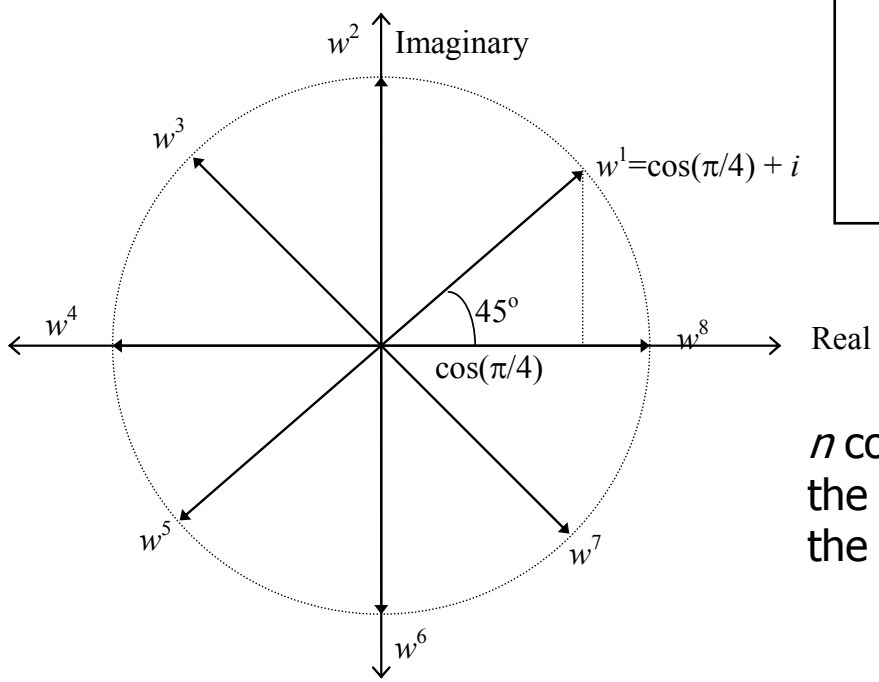
- Example: The complex number $e^{2pi/n}$ is a primitive n-th root of unity, where

Check: if properties are satisfied

$$1.\ \ \omega^1 = e^{\frac{2\pi i}{n}} \neq 1$$

$$2.\ \ \omega^n = \left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

$$3.\ \ \mathrm{S} = \sum_{p=0}^{n-1} \omega^{jp} = \omega^0 + \omega^j + \omega^{2j} + \omega^{3j} + \ldots\ldots + \omega^{j(n-1)} = 0$$



n complex roots of unity equally spaced around the circle of unit radius centered at the origin of the complex plane.

Fast Fourier Transform

- Presented by Cooley and Tukey in 1965, but invented several times, including by Gauss (1809) and Danielson & Lanczos (1948)
- Danielson Lanczos lemma

$$
\begin{aligned}
F_k &= \sum_{j=0}^{N-1} e^{2\pi i j k/N} f_j \\
&= \sum_{j=0}^{N/2-1} e^{2\pi i k(2j)/N} f_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi i k(2j+1)/N} f_{2j+1} \\
&= \sum_{j=0}^{N/2-1} e^{2\pi i k j/(N/2)} f_{2j} + W^k \sum_{j=0}^{N/2-1} e^{2\pi i k j/(N/2)} f_{2j+1} \\
&= F_k^e + W^k \, F_k^o
\end{aligned}
$$

- So far we have seen what happens on the right hand side
- How about the left hand side?
- When we split the sums in two we have two sets of sums with $N/2$ quantities for N points.
- So the complexity is $N^2/2 + N^2/2 = N^2$
- So there is no improvement

- Need to reduce the sums on the left hand side as well
  - We need to reduce the number of sums computed from 2N to a lower number
  - Notice that the values corresponding to k and $k+N/2$ will be the same.
  - The transforms F_e^k and F_o^k are periodic in k with length $N/2$.
  - So we need only compute half of them!

FFT

- So DFT of order *N* can be expressed as sum of two DFTs of order *N/2 evaluated at N/2* points
- Does this improve the complexity?
- Yes $(N/2)^2+(N/2)^2 = N^2/2 < N^2$
- But we are not done ….
- Can apply the lemma recursively

$$F_k^e = F_k^{ee} + W^k F_k^{eo}, \quad F_k^o = F_k^{oe} + W^k F_k^{oo},$$

- Finally we have a set of one point transforms
- One point transform is identity

$$F_k^{eoeeoeo\cdots oee} = f_n$$

FFT Algorithm

FFT $(n, a_0, a_1, a_2, \ldots, a_{n-1})$

```
if (n == 1)                    // n is a power of 2
   return a_0

ω ← e^(2π i / n)
(e_0,e_1,e_2,...,e_n/2-1) ← FFT(n/2, a_0,a_2,a_4,...,a_n-2)
(d_0,d_1,d_2,...,d_n/2-1) ← FFT(n/2, a_1,a_3,a_5,...,a_n-1)

for k = 0 to n/2 - 1
      y_k ← e_k + ω^k d_k
   y_k+n/2 ← e_k - ω^k d_k

return (y_0,y_1,y_2,...,y_n-1)
```

O(n) complex multiplies if we pre-compute ω^k.

$$T(n) = 2T(n/2) + O(n) \;\Rightarrow\; T(n) = O(n \log n)$$

Complexity

- Each F_k is a sum of $\log_2$ N transforms and (factors)
- There are N F_k s
- So the algorithm is $O(N \log_2 N)$
- *This is a recursive algorithm*

FFTtx

```
function y = ffttx(x)
%FFTTX  Textbook Fast Finite Fourier Transform.
%   FFTTX(X) computes the same finite Fourier transform as FFT(X).
%   The code uses a recursive divide and conquer algorithm for
%   even order and matrix-vector multiplication for odd order.
%   If length(X) is m*p where m is odd and p is a power of 2, the
%   computational complexity of this approach is O(m^2)*O(p*log2(p)).

x = x(:);
n = length(x);
omega = exp(-2*pi*i/n);
if rem(n,2) == 0
   % Recursive divide and conquer
   k = (0:n/2-1)';
   w = omega .^ k;
   u = ffttx(x(1:2:n-1));
   v = w.*ffttx(x(2:2:n));
   y = [u+v; u-v];
else
   % The Fourier matrix.
   j = 0:n-1;
   k = j';
   F = omega .^ (k*j);
   y = F*x;
end
```