Outline

- Notation and Basics
- Motivation
- Linear systems of equations
  - Gauss Elimination, LU decomposition
- Linear Spaces and Operators
  - Addition, scalar multiplication, scalar product, transformation, operator, basis
- Eigenvalues, Eigenvectors
- Solvability conditions (“alternative theorem”)
  - Adjoint, null space, orthogonality

Outline

- Euclidean space R³
  - distance, angles, rotations
- Metric Space
  - Distance, angles, rotations
- Least Squares
- Singular Value Decomposition
- Other Matrix decompositions

Motivation

- Fundamental to representation and numerical solution of almost all problems including those in vision and computational statistics.
  - Solving equations for calibration, stereo, tracking, …
- Geometry is fundamental to vision. However one way of doing geometry is via algebra.
  - Intersections of lines, points, planes. Determining angles. Determining orthogonal projections …
- Modern computer vision is formulated in terms of “projective geometry”. Most results in projective geometry are stated algebraically and require knowledge of concepts such as rank, null space, constraints

Applications

- Rectification of images
- Calibrating cameras
- Transforming color spaces
- Tracking motion of a rigid body
- Applying constraints from multiple views
- Parametrizing fundamental matrix and trifocal tensor.

Vectors

- A vector \( \mathbf{x} \) of dimension \( d \) represents a point in a \( d \) dimensional space
- Examples
  - A point in 3D Euclidean space \([x,y,z]\) or 2D image space \([u,v]\)
  - A point in a projective space \(P^3[U,V,W]\) or in projective space \(P^2[U,V,W]\)
  - Point in color space \([r,g,b]\) or \([u, v]\)
  - Point in an infinite dimensional functional space on a Fourier basis
  - Vector of intrinsic parameters for a camera (focal length, skew ratio, …)
- Essentially a short-hand notation to denote a grouping of points
  - No special structure yet
Vectors and Matrices

• $d$ dimensional column vector $\mathbf{x}$ and its transpose $\mathbf{x}' = (x_1, x_2, \ldots, x_d)$

• $d\times n$ dimensional matrix $\mathbf{M}$ and its transpose $\mathbf{M}'$

Determinant

• Determinant of a $2\times 2$ matrix $m_{11}m_{22} - m_{12}m_{21}$

• For a higher dimensional matrix we have a recursive definition

\[
\text{Determinant} = \begin{vmatrix}
\begin{array}{ccc}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{array}
\end{vmatrix} = m_{11}(m_{22}m_{33} - m_{23}m_{32}) - m_{12}(m_{21}m_{33} - m_{23}m_{31}) + m_{13}(m_{21}m_{32} - m_{22}m_{31})
\]

Determinant: Remarks

• Determinant determines “magnitude” of matrix. Matrix with determinant $=0$ is called singular.

• Determinant is important in theorems

• Practically the way to compute the determinant is not this way.

• Homework problem -- determine number of operations for recursive algorithm.

Matrix basics

• Square matrix: number of rows = number of columns

• Symmetric matrix $A = A'$

• Skew symmetric matrix $A' = -A$

• Identity $\mathbf{I} = \mathbf{I}'$

• Lower triangular

\[
\begin{bmatrix}
a & 0 & \cdots & 0 \\
b & c & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
f & g & \cdots & d
\end{bmatrix}
\]

• Upper triangular

\[
\begin{bmatrix}
a & c & \cdots & d \\
0 & b & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & e
\end{bmatrix}
\]

Matrix vector product

• $m\times n$ dimensional matrix $\mathbf{M}$ multiplies by an $n$ dimensional vector $\mathbf{x}$ to produce a $m$ dimensional vector

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\]

\[
\mathbf{M} = \begin{pmatrix}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{m1} & m_{m2} & \cdots & m_{mn}
\end{pmatrix}
\]

\[
\mathbf{M}' = \begin{pmatrix}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{m1} & m_{m2} & \cdots & m_{mn}
\end{pmatrix}
\]

Linear systems of equations

• Systems of equations

\[
a_1x_1 + a_2x_2 + a_3x_3 = b_1
\]

• Can be written as

\[
a_1x_1 + a_2x_2 + a_3x_3 = b_1
\]

• Can change or scale rows

\[
\text{Ax} = \mathbf{b}
\]

• Solved via Gauss elimination

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

• Reduce system to product of lower or upper triangular matrix and $\mathbf{x}$

• $O(N^3)$ operations to solve triangular system

• $O(N^2)$ operations to perform Gauss elimination
LU decomposition
- Any matrix can be written as a product of a lower triangular and upper triangular matrix.
- Most used algorithm in linear algebra.
  $$\mathbf{A} = \mathbf{LU}$$
- Practically implemented by reordering equations and scaling them so that loss of accuracy is minimized.
  $$\mathbf{A} = \mathbf{PLU}$$
- Scaled LU decomposition with “partial pivoting”
- When \( \mathbf{A} \) is symmetric positive definite “Cholesky decomposition”
  $$\mathbf{A} = \mathbf{LL}^T$$

Linear/Vector Spaces
- Previous stuff was somewhat mechanical.
- In vision we have to answer questions when
  - Models provide equations that are singular or degenerate. What can we say about the solutions? Can we restrict them?
  - Number of unknowns may be more or less than the number of observations. Can we still obtain a meaningful solution?
  - How “far” is an approximation from a solution? How do we measure this distance?
  - Matrices are operators that take one vector into another. What can we say about the properties of the operator? When is an equation involving an operator solvable?

Operators
- Function, Transformation, Operator, Mapping: synonyms
- A function takes elements \( x \) defined on its “Domain” \( D \) to elements \( y \) in its “Range” \( R \) which is part of \( E \)
- If for each \( y \) in \( R \) there is exactly one \( x \) in \( D \) the function is one-to-one. In this case an inverse exists whose domain is \( R \) and whose range is \( D \)
- We are interested in situations where \( R \) and \( D \) are finite-dimensional linear spaces

Dependence and dimensionality
- A set of vectors is dependent if for some scalars \( \alpha_1, \ldots, \alpha_k \) not all zero we can write
  $$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k = \mathbf{0}$$
- Otherwise the vectors are independent.
  - If the zero vector is part of a set of vectors that set is dependent. If a set of vectors is dependent so is any larger set which contains it.
  - A linear space is \( n \) dimensional if it possesses a set of \( n \) independent vectors but every \( n+1 \) dimensional set is dependent.
  - A set of vectors \( \mathbf{b}_1, \ldots, \mathbf{b}_k \) is a basis for a \( k \) dimensional space \( X \) if each vector in \( X \) can be expressed in one and only one way as a linear combination of \( \mathbf{b}_1, \ldots, \mathbf{b}_k \)
  - One example of a basis are the vectors \((1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,0,\ldots,1)\)

Distances/Metrics and Norms
- We would like to measure distances and directions in the vector space the same way that we do it in Euclidean 3D
- Distance function \( d(\mathbf{u},\mathbf{v}) \) makes a vector space a metric space if it satisfies
  - \( d(\mathbf{u},\mathbf{v}) > 0 \) for \( \mathbf{u} \neq \mathbf{v} \) different
  - \( d(\mathbf{u},\mathbf{u}) = 0 \)
  - \( d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u}) \) (triangle inequality)
- Norm ("length")
  - \( \| \mathbf{u} \| = 0 \) for \( \mathbf{u} = \mathbf{0} \)
  - \( \| \mathbf{u} \| = |\mathbf{u}| \) for \( \mathbf{u} \neq \mathbf{0} \)
  - \( \| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \| \)
- Normed linear space is a metric space with the metric defined by \( d(\mathbf{u},\mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| \) and \( \| \mathbf{u} \| = d(\mathbf{u},\mathbf{0}) \)
Dot Product

- Dot product of two vectors with same dimension $\langle x, y \rangle = x'y = \sum x_i y_i$
- Dot product space behaves like Euclidean $\mathbb{R}^3$
- Dot product defines a norm and a metric.
  - Parallelogram law $||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$
- Orthogonal vectors $\langle u, v \rangle = 0$
- Angle between vectors $\cos \theta = \frac{\langle x, y \rangle}{||x|| ||y||}$
- Orthonormal basis -- elements have norm 1 and are perpendicular to each other
- Other distances and products can also define a space:
  - Mahalanobis distance

Matrices as operators

- Matrix is an operator that takes a vector to another vector.
  - Square matrix takes it to a vector in the space of the same dimension.
- Dot product provides a tool to examine matrix properties
  - Adjoint matrix $\langle Au, v \rangle = \langle u, A'v \rangle$
  - Square Matrix fully defined as result of its operation on members of a basis.
    $A_{ij} = \langle Ab_j, b_i \rangle$

Eigenvalues and Eigenvectors

- Square matrix possesses its own natural basis.
- Eigen relation $Au = \lambda u$
- Matrix $A$ acts on vector $u$ and produces a scaled version of the vector.
- Eigen is a German word meaning “proper” or “specific”
- $u$ is the eigenvector while $\lambda$ is the eigenvalue.
  - If $u$ is an eigenvector then $u = 0$
  - If $||u|| = 1$ then we call it a normal eigenvector
  - $\lambda$ is like a measure of the “strength” of $A$ in the direction of $u$
- Set of all eigenvalues and eigenvectors of $A$ is called the “spectrum of $A$”

Motivation: Stereo

- Point $(x, y)$ on the image plane lies on a line in the world that passes through the image point and center of projection
- Image of this line in the world will form a line in another camera

Epipolar Constraint

- Point in one image lies on the “epipolar line” in the other image
- Algebraic statement of geometry
  - Equation of line in the other image is $Fm$
  - Condition that the point $m'$ lies on this line is $m'Fm = 0$
- $F$ is the “fundamental matrix”
- Estimating the fundamental matrix is an important problem in vision

Eight point algorithm: Determining the Fundamental matrix

- Given a set of matching points in the images, Determine $F$

$$
\begin{bmatrix}
0 & f_{11} & f_{12} & f_{13} \\
0 & f_{21} & f_{22} & f_{23} \\
0 & f_{31} & f_{32} & f_{33}
\end{bmatrix}
\begin{bmatrix}
f_{11} & f_{12} & f_{13} & 0 \\
f_{21} & f_{22} & f_{23} & 0 \\
f_{31} & f_{32} & f_{33} & 0
\end{bmatrix} = 0
$$

$$(f_{1i} + f_{2i} + f_{3i}) + (f_{1i} + f_{2i} + f_{3i}) = 0$$
Determining F

- Write expression as an equation in the unknown elements.

- If we have eight points we can solve for elements of F, (e.g. via LU)

  \[
  \begin{bmatrix}
  \mu' u' & \mu' v' & \mu' & \nu' u' & \nu' v' & \nu' & 1 \end{bmatrix}
  \begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3
  \end{bmatrix} = [-1]
  \]

- If we have more than eight points we can use a least squares formulation.