Euclidean versus Projective Geometry

- Euclidean geometry describes shapes “as they are”
  - Properties of objects that are unchanged by rigid motions
    - Lengths
    - Angles
    - Parallelism
- Projective geometry describes objects “as they appear”
  - Lengths, angles, parallelism become “distorted” when we look at objects
  - Mathematical model for how images of the 3D world are formed.

Overview

- Tools of algebraic geometry
- Informal description of projective geometry in a plane
- Descriptions of lines and points
- Points at infinity and line at infinity
- Projective transformations, projectivity matrix
- Example of application
- Special projectivities: affine transforms, similarities, Euclidean transforms
- Cross-ratio invariance for points, lines, planes

Tools of Algebraic Geometry 1

- Plane passing through origin and perpendicular to vector $\mathbf{n} = (a, b, c)$ is locus of points $\mathbf{x} = (x_1, x_2, x_3)$ such that $\mathbf{n} \cdot \mathbf{x} = 0$
- Plane through origin is completely defined by $(a, b, c)$

Tools of Algebraic Geometry 2

- A vector parallel to intersection of 2 planes $(a, b, c)$ and $(a', b', c')$ is obtained by cross-product

Tools of Algebraic Geometry 3

- Plane passing through two points $\mathbf{x}$ and $\mathbf{x}'$ is defined by

\[ (a, b, c) = \mathbf{x} \times \mathbf{x}' \]
**Projective Geometry in 2D**

- We are in a plane \( P \) and want to describe lines and points in \( P \).
- We consider a third dimension to make things easier when dealing with infinity.
  - Origin \( O \) out of the plane, at a distance equal to 1 from plane.
  - To each point \( m \) of the plane \( P \) we can associate a single ray \( \mathbf{x}(x_1, x_2, x_3) \).
  - To each line \( L \) of the plane \( P \) we can associate a single plane \((a, b, c)\).

\[
L = (a, b, c)
\]

\[
\mathbf{x} = (x_1, x_2, x_3)
\]

From Projective Plane to Euclidean Plane

- How do we “land” back from the projective world to the 2D world of the plane?
  - For point, consider intersection of ray \( \mathbf{x}(x_1, x_2, x_3) \) with plane \( \mathbf{x}'(x_1', x_2', x_3') \) and \( \mathbf{x}(x_1, x_2, x_3) \).
  - For line, intersection of plane \( L = (a, b, c) \) and \( L' = (a', b', c') \).

Ideal Points and Line at Infinity

- The points \( x = (x_1, x_2, 0) \) do not correspond to finite points in the plane. They are points at infinity, also called ideal points.
  - The line \( L = (0,1,0) \) passes through all points at infinity, since \( L \cdot x = 0 \).
  - Two parallel lines \( L = (a, b, c) \) and \( L' = (a', b', c') \) intersect at the point \( x = L \\
  = (a, b, c) \).
  - Any line \( m, n \) intersects the line at infinity at \((b, a, 0)\). So the line at infinity is the set of all points at infinity.

**Properties**

- Point \( X \) belongs to line \( L \) if \( L \cdot X = 0 \).
- Equation of line \( L \) in projective geometry is \( a_1 x_1 + b_1 x_2 + c_1 x_3 = 0 \).
- We obtain homogeneous equations

\[
L = (a, b, c)
\]

\[
\mathbf{x} = (x_1, x_2, x_3)
\]

**Lines and Points**

- Two lines \( L = (a, b, c) \) and \( L' = (a', b', c') \) intersect in the point \( x = L \\
  = (a, b, c) \).
- The line through 2 points \( x \) and \( x' \) is \( L = x \\
  = x' \).
- Duality principle: To any theorem of 2D projective geometry, there corresponds a dual theorem, which may be derived by exchanging the roles of points and lines in the original theorem.

\[
L = (a, b, c)
\]

\[
\mathbf{x} = (x_1, x_2, x_3)
\]
Ideal Points and Line at Infinity

- With projective geometry, two lines always meet in a single point, and two points always lie on a single line.
- This is not true of Euclidean geometry, where parallel lines form a special case.

Projective Transformations in a Plane

- **Projectivity**
  - Mapping from points in plane to points in plane
  - 3 aligned points are mapped to 3 aligned points
- Also called
  - **Collineation**
  - **Homography**

Projectivity Theorem

- A mapping is a **projectivity** if and only if the mapping consists of a linear transformation of homogeneous coordinates \( \mathbf{x}' = \mathbf{Hx} \) with \( \mathbf{H} \) non-singular
- **Proof**:
  - If \( x_1, x_2, x_3 \) are 3 points that lie on a line \( \mathbf{L} \), and \( x_1', x_2', x_3' \) lie on a line \( \mathbf{L}' \)
    - \( \mathbf{L'}x = 0, \mathbf{L'H}^{-1} \mathbf{Hx} = 0 \), so points \( \mathbf{Hx} \) lie on line \( \mathbf{H^{-1}L} \)
- Converse is hard to prove, namely if all collinear sets of points are mapped to collinear sets of points, then there is a single linear mapping between corresponding points in homogeneous coordinates

Projectivity Matrix

- The matrix \( \mathbf{H} \) can be multiplied by an arbitrary non-zero number without altering the projective transformation
- Matrix \( \mathbf{H} \) is called a "homogeneous matrix" (only ratios of terms are important)
- There are 8 independent ratios. It follows that projectivity has 8 degrees of freedom
- A projectivity is simply a linear transformation of the rays

Examples of Projective Transformations

- Central projection maps **planar scene** points to image plane by a projectivity
  - True because all points on a scene line are mapped to points on its image line
- The image of the same planar scene from a second camera can be obtained from the image from the first camera by a projectivity
  - True because \( x_1' = \mathbf{H}x_1, x_2' = \mathbf{H}x_2 \)
  - \( \mathbf{H} = \) so \( x_3' = \mathbf{H}x_3 \)

Computing Projective Transformation

- Since matrix of projectivity has 8 degrees of freedom, the mapping between 2 images can be computed if we have the coordinates of 4 points on one image, and know where they are mapped in the other image
  - Each point provides 2 independent equations
    - \( x_1' = \frac{h_1x + h_2y + h_3}{h_4x + h_5y + h_6} \)
    - \( y_1' = \frac{h_7x + h_8y + h_9}{h_{10}x + h_{11}y + h_{12}} \)
  - Equations are linear in the 8 unknowns \( h_{ij} \)
Example of Application

- Robot going down the road
- Large squares painted on the road to make it easier
- Find road shape without perspective distortion from image
  - Use corners of squares: coordinates of 4 points allow us to compute matrix $H$
  - Then use matrix $H$ to compute 3D road shape

Special Projectivities

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Projective Geometry

Projective Space $P_n$

- A point in a projective space $P_n$ is represented by a vector of $n+1$ coordinates.
- At least one coordinate is non-zero.
- Coordinates are called homogeneous or projective coordinates.
- Vector $x$ is called a coordinate vector.
- Two vectors $x = (x_1, x_2, \ldots, x_{n+1})$ and $y = (y_1, y_2, \ldots, y_{n+1})$ represent the same point if and only if there exists a scalar $l$ such that $l x = y$.

Projectivity in 1D

- A projective transformation of a line is represented by a 2x2 matrix.
- Transformation has 3 degrees of freedom corresponding to the 4 elements of the matrix, minus one for overall scaling.
- Projectivity matrix can be determined from 3 corresponding points.

Cross-Ratio Invariance in 1D

- Cross-ratio of 4 points $A, B, C, D$ on a line is defined as $\text{Cross}(A,B,C,D) = \frac{BC}{BA} \cdot \frac{AD}{AB}$.
- Cross-ratio is not dependent on which particular homogeneous representation of the points is selected: scales cancel between numerator and denominator. For $A = (a, 1)$, $B = (b, 1)$, etc. we get $\text{Cross}(A,B,C,D) = \frac{CD}{CA} \cdot \frac{BA}{BC}$.
- Cross-ratio is invariant under any projectivity.

Invariants

- Collinearity, Cross-ratios
- Parallelism, Ratios of areas, Length ratios
- Angles, Lengths, Areas
Cross-Ratio Invariance in 1D

- For the 4 sets of collinear points in the figure, the cross-ratio for corresponding points has the same value.

Cross-Ratio Invariance between Lines

- The cross-ratio between 4 lines forming a pencil is invariant when the point of intersection C is moved.
- It is equal to the cross-ratio of the 4 points.

Projective Geometry in 3D

- Space \( P_3 \) is called projective space.
- A point in 3D space is defined by 4 numbers \((x_1, x_2, x_3, x_4)\).
- A plane is also defined by 4 numbers \((u_0, u_1, u_2, u_3)\).
- Equation of plane is \( \sum u_i x_i = 0 \).
- The plane at infinity is the plane \((0,0,0,1)\). Its equation is \( x_4 = 0 \).
- The points \((x_0, x_2, x_3, 0)\) belong to that plane in the direction \((x_1, x_2, x_3)\) of Euclidean space.
- A line is defined as the set of points that are a linear combination of two points \(P_1, P_2\).
- The cross-ratio of 4 planes is equal to the cross-ratio of the lines of intersection with a fifth plane.

Central Projection

If world and image points are represented by homogeneous vectors, central projection is a linear mapping between \(P_3\) and \(P_2\):

\[
\begin{align*}
X_i &= \frac{x_i}{z_i} \\
Y_i &= \frac{y_i}{z_i}
\end{align*}
\]

If world and image points are represented by homogeneous vectors, central projection is a linear mapping between \(P_3\) and \(P_2\):

\[
\begin{bmatrix}
x_i \\
y_i \\
z_i
\end{bmatrix} =
\begin{bmatrix}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_s \\
y_s \\
z_s
\end{bmatrix}
\]

References: