Computing the factorization

- QR is useful … so how do we factorize a matrix $A$?
- In LU we computed a upper triangular matrix by computing adding multiples of other rows so that elements below a given column were zeroed out
- The multipliers were stored in $L$ which gave us $A = LU$
- Here we want to zero out entries below the diagonal but do it with orthogonal matrices
- Two strategies
  - Zero out a column at a time using a matrix $Q_1$ so that $Q_1^T A$ gives us all entries below a certain one in a column as zero
    - Householder transformations
    - Result $Q_n^T \ldots Q_2^T Q_1^T A = R$ or $A = Q_1 \ldots Q_{n-1} Q_n R = Q R$
  - Zero out one specific entry of a column at a time
    - Givens rotations
- Product of orthogonal matrices is orthogonal
To compute QR

- Perform a sequence of orthogonal transformations that zero out elements
- Orthogonal transformations can be rotations or reflections or combinations
- Givens Rotation:
  - Givens matrix has elements
  - \( c^2 + s^2 = 1 \)
  - How do we use a rotation to zero out an element?
  - Let \( z = [z_1, z_2]^t \)
  - We want to eliminate \( z_2 \)
  - Eliminate \( z_2 \) \( (c^2 + s^2)z_1 = cx \), \( c = z_1 / x \).
  - Similarly we get \( s = z_2 / x \), and \( z_1^2 + z_2^2 = x^2 \)

\[
G = \begin{bmatrix} c & s \\ s & -c \end{bmatrix}
\]
Givens QR

• To apply idea to larger matrix, embed the Givens matrix in identity matrix. We will use the notation $G_{ij}$ to denote an $n \times n$ identity matrix with its $i$th and $j$th rows modified to include the Givens rotation: for example, if $n = 6$, then

$$G_{25} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & c & 0 & 0 & s & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & s & 0 & 0 & -c & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

and multiplication of a vector by this matrix leaves all but rows 2 and 5 of the vector unchanged.

• **Algorithm**

  for $i=1, ..., n$

  for $j=i+1, ..., m$

  Find Givens matrix $G_{ij}$ to zero out $j,i$ element of $A$ using the value at position $(i,i)$

  $A = G_{ij}A$

  end

end

Goal: reflect axial vector through a (hyper)-plane

- Axial vector is any vector through the origin. Let it be \( \mathbf{u} \)

- Goal reflect it through a plane
- Let \( \mathbf{a} \) be the unit vector normal to the plane
- \( \mathbf{u} \cdot \mathbf{a} = \|\mathbf{u}\| \cos \theta \)
- To get reflected vector in the plane
  Subtract twice the component of \( \mathbf{u} \) along \( \mathbf{a} \)
- \( \mathbf{v} = \mathbf{u} - 2 \mathbf{a} \|\mathbf{u}\| \cos \theta \)
- \( \cos \theta = (\mathbf{a} \cdot \mathbf{u})/(\|\mathbf{u}\|) \)
Householder transform

• Achieve this reflection via multiplication by an orthogonal matrix

• \( v = Qu = u - 2 a \|u\| \cos \theta \)

\[
= u - 2 a \|u\| (a \cdot u) / (\|u\|)
\]

= \((I - 2 \ a a^t)u\)

• What if \( a \) is not a unit vector?

\((I - 2 (aa^t)/(a^t a))u\)

• \( Q = I - 2 (aa^t)/(a^t a) \)
Householder Transformations

The *Householder transformation* determined by vector $v$ is:

$$H = I - 2 \frac{vv^T}{v^Tv}$$

outer product, $n \times n$ matrix

inner product, scalar

To apply it to a vector $x$, compute:

$$Hx = \left( I - 2 \frac{vv^T}{v^Tv} \right)x = x - 2 \frac{v(v^Tx)}{v^Tv}$$

$$Hx = x - \left( 2 \frac{v^Tx}{v^Tv} \right)v$$

scalar
Householder Geometry

- $Hx$ is $x$ reflected through the hyperplane perpendicular to $v$ ($p : p^Tv=0$)
Householder Properties

- $H$ is symmetric, since

$$H^T = \left( I - 2 \frac{vv^T}{v^Tv} \right)^T = I^T - 2 \left( \frac{vv^T}{v^Tv} \right) = I - 2 \frac{v^Tv}{v^Tv} = H$$

- $H$ is orthogonal, since

$$H^T H = H \overline{H} = \left( I - 2 \frac{vv^T}{v^Tv} \right) \left( I - 2 \frac{vv^T}{v^Tv} \right)$$

$$= I - 4 \frac{vv^T}{v^Tv} + 4 \frac{v(v^Tv)v^T}{(v^Tv)^2} = I - 4 \frac{vv^T}{v^Tv} + 4 \frac{vv^T}{v^Tv} = I$$

and $H^T H = I$ implies $H^T = H^{-1}$
Householder to Zero Matrix Elements

We’ll use Householder transformations to zero subdiagonal elements of a matrix.

Given any vector \( a \), find the \( v \) that determines an \( H \) such that,

\[
Ha = \alpha e_1 = \alpha [1, 0, 0, ..., 0]^T
\]

Now solve for \( v \):

\[
Ha = a - \left( 2 \frac{v^T a}{v^T v} \right) v = a - \mu v = \alpha e_1
\]

where \( \mu \) is parenthesized scalar, related to length of \( v \)

\[
\Rightarrow \quad v = \frac{(a - \alpha e_1)}{\mu}
\]

We're free to choose \( \mu = 1 \), since \( ||v|| \) does not affect \( H \)
Choosing the Vector $\nu$

So $\nu = a - \alpha e_1$ for some scalar $\alpha$.

But $\|Ha\|_2 = \|a\|_2$

(prove this by expanding $\|Ha\|_2^2 = (Ha)^T Ha$)

and $\|Ha\|_2 = |\alpha|$ by design, so $\alpha = \pm \|a\|_2$

(either sign will work).

To avoid $\nu \approx 0$ we choose $\alpha = -\text{sign}(a_1)\|a\|_2$, so $\nu = a + \text{sign}(a_1)\|a\|_2 e_1$ is our answer.
Applying Householder Transforms

• Don’t compute $Hx$ explicitly, that costs $3n^2$ flops.
• Instead use the formula given previously,

$$Hx = x - \left(2 \frac{v^T x}{v^T v}\right)v$$

which costs $4n$ flops (if you pre-compute $v^Tv$ or pre-normalize $v^Tv=2$).

• Typically, when using Householder transformations, you never compute the matrix $H$; it’s only used in derivation and analysis.
QR Decomposition

- Householder transformations are a good way to zero out subdiagonal elements of a matrix.
- $A$ is decomposed:
  \[
  Q^T A = \begin{bmatrix} R \\ 0 \end{bmatrix} \quad \text{or} \quad Q Q^T A = A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}
  \]
- where $Q^T = H_n \ldots H_2 H_1$ is the orthogonal product of Householders and $R$ is upper triangular.
- Overdetermined system $A x = b$ is transformed into the easy-to-solve
  \[
  \begin{bmatrix} R \\ 0 \end{bmatrix} x = Q^T b
  \]