Computing the factorization

- QR is useful … so how do we factorize a matrix A?
- In LU we computed a upper triangular matrix by computing adding multiples of other rows so that elements below a given column were zeroed out
- The multipliers were stored in L which gave us A=LU
- Here we want to zero out entries below the diagonal but do it with orthogonal matrices
- Two strategies
- Zero out a column at a time using a matrix Q_1 so that Q^t_1 A gives us all entries below a certain one in a column as zero
  - Householder transformations
  - Result $Q^t_n \dots Q^t_2 Q^t_1 A = R$ or $A = Q_1 \dots Q_{n-1} Q_n R = Q R$
- Zero out one specific entry of a column at a time
  - Givens rotations
- Product of orthogonal matrices is orthogonal

To compute QR

- Perform a sequence of orthogonal transformations that zero out elements
- Orthogonal transformations can be rotations or reflections or combinations
- Givens Rotation:
- Givens matrix has elements

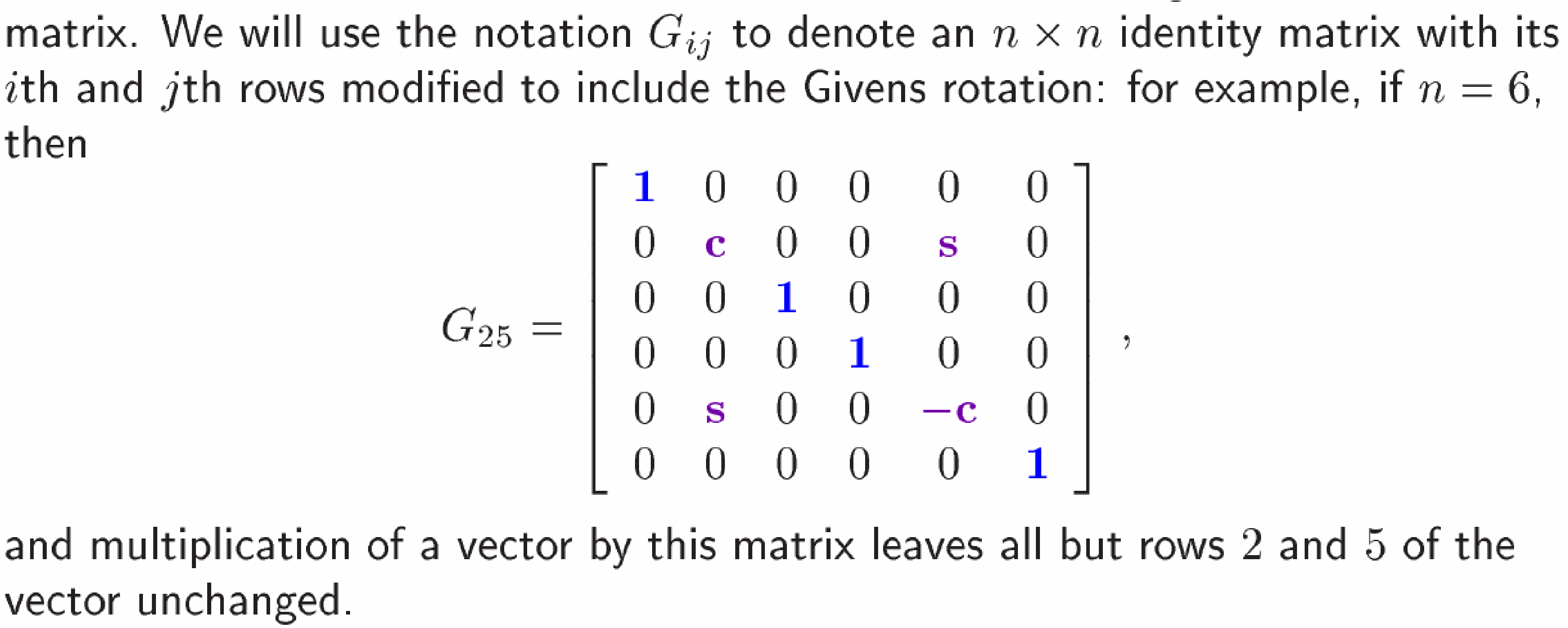
$$G = \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$$

- $c^2+s^2=1$
- How do we use a rotation to zero out an element?
- Let z= $[z_1\ z_2]^t$
- We want

$$Gz = \begin{bmatrix} cz_1 + sz_2 \\ sz_1 - cz_2 \end{bmatrix} = xe_1$$

- Eliminate z_2 $\quad (c^2 + s^2)z_1 = cx\,, \quad c = z_1/x\,.$
- Similarly we get $s=z_2/x,$ and $z_1^2+z_2^2 = x^2$

Givens QR



- To apply idea to larger matrix, embed the Givens matrix in identity matrix. We will use the notation G_{ij} to denote an $n \times n$ identity matrix with its ith and jth rows modified to include the Givens rotation: for example, if $n = 6$, then

$$G_{25} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{c} & 0 & 0 & \mathbf{s} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \mathbf{s} & 0 & 0 & -\mathbf{c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

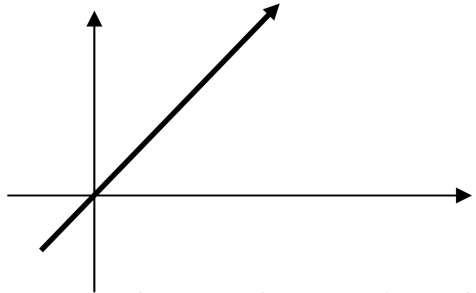
and multiplication of a vector by this matrix leaves all but rows 2 and 5 of the vector unchanged.

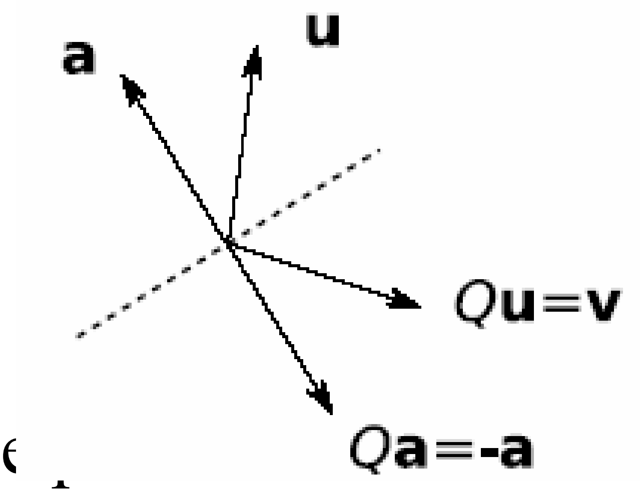
- Algorithm

```
for $i=1, ..., n$
    for $j=i+1, ..., m$
        Find Givens matrix $G_{ij}$ to zero out $j,i$ element of A
        using the the value at position $(i,i)$
        $A=G_{ij}A$
    end
end
```

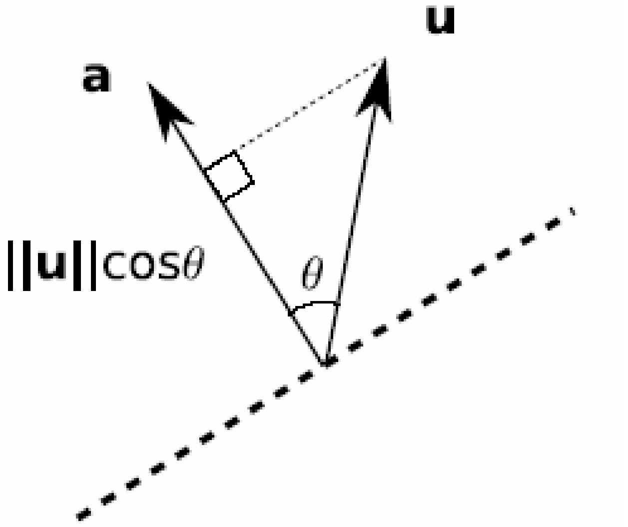
Goal: reflect axial vector through a (hyper)-plane

- Axial vector is any vector through the origin. Let it be **u**





- Goal reflect it through a plane
- Let **a** be the unit vector normal to the
- $\mathbf{u}\cdot\mathbf{a} = \|\mathbf{u}\| \cos\theta$
- To get reflected vector in the plane Subtract twice the component of **u** along **a**
- $\mathbf{v} = \mathbf{u} - 2\,\mathbf{a}\|\mathbf{u}\| \cos\theta$
- $\cos\theta = (\mathbf{a} \cdot \mathbf{u})/(\|\mathbf{u}\|)$



Householder transform

- Achieve this reflection via multiplication by an orthogonal matrix
- $\mathbf{v}=\mathbf{Qu}= \mathbf{u} - 2\,\mathbf{a}\|\mathbf{u}\| \cos\theta$

□ $= \mathbf{u} - 2\,\mathbf{a}\|\mathbf{u}\|\,(\mathbf{a} \cdot \mathbf{u})/(\|\mathbf{u}\|)$

- $= (\mathbf{I} - 2\,\mathbf{aa}^t)\mathbf{u}$
- What if **a** is not a unit vector?

$(\mathbf{I} - 2\,(\mathbf{aa}^t)/(\mathbf{a}^t\,\mathbf{a}))\mathbf{u}$

- $\mathbf{Q}=\mathbf{I} - 2\,(\mathbf{aa}^t)/(\mathbf{a}^t\,\mathbf{a})$

Householder Transformations

The *Householder transformation* determined by vector v is:

$$H = I - 2\frac{vv^T}{v^T v}$$

outer product, n×n matrix

inner product, scalar

To apply it to a vector x, compute:

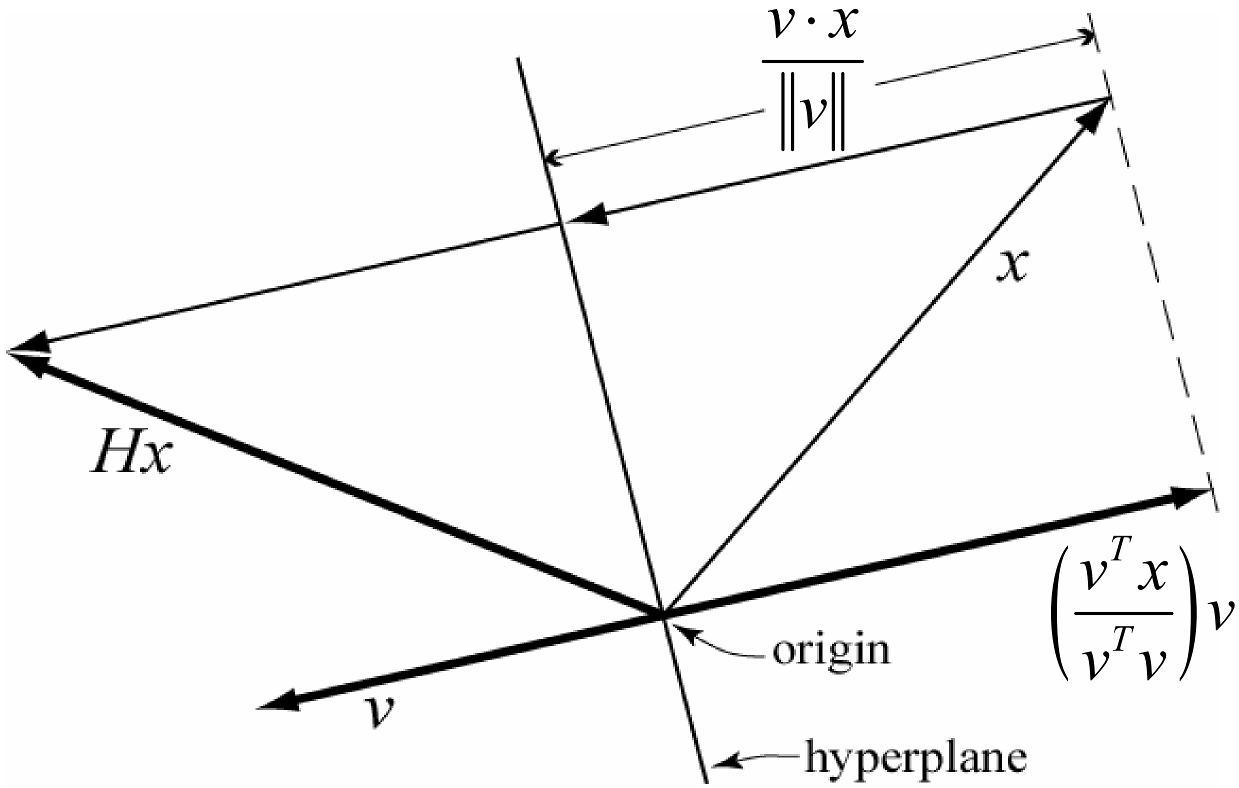
$$Hx = \left(I - 2\frac{vv^T}{v^T v} \right) x = x - 2\frac{v(v^T x)}{v^T v}$$

$$\boxed{Hx = x - \left(2\frac{v^T x}{v^T v} \right) v}$$

scalar

Householder Geometry

- ***Hx* is *x* reflected through the hyperplane perpendicular to *v*** ($p : p^{\mathrm{T}}v$=0)



Householder Properties

- H is symmetric, since

$$H^T = \left(I - 2\frac{vv^T}{v^T v} \right)^T = I^T - 2\frac{(vv^T)^T}{v^T v} = I - 2\frac{v^{T^T} v^T}{v^T v} = H$$

- H is orthogonal, since

$$H^T H = HH = \left(I - 2\frac{vv^T}{v^T v} \right)\left(I - 2\frac{vv^T}{v^T v} \right)$$

$$= I - 4\frac{vv^T}{v^T v} + 4\frac{v(v^T v)v^T}{(v^T v)^2} = I - 4\frac{vv^T}{v^T v} + 4\frac{vv^T}{v^T v} = I$$

and $H^T H = I$ implies $H^T = H^{-1}$

Householder to Zero Matrix Elements

We’ll use Householder transformations to zero subdiagonal elements of a matrix.

Given any vector *a*, find the *v* that determines an *H* such that,

$$Ha = \alpha e_1 = \alpha[1, 0, 0, ..., 0]^T$$

Now solve for *v*:

$$Ha = a - \left(2\frac{v^T a}{v^T v} \right) v = a - \mu v = \alpha e_1$$

where μ is parenthesized scalar, related to length of v

$$\Rightarrow \; v = (a - \alpha e_1) / \mu$$

We're free to choose $\mu = 1$, since $\|v\|$ does not affect H

Choosing the Vector v

So $v = a - \alpha e_1$ for some scalar α.

But $\|Ha\|_2 = \|a\|_2$

(prove this by expanding $\|Ha\|_2^2 = (Ha)^T Ha$)

and $\|Ha\|_2 = |\alpha|$ by design, so $\alpha = \pm\|a\|_2$

(either sign will work).

To avoid $v \approx 0$ we choose $\alpha = -\text{sign}(a_1)\|a\|_2$,

so $v = a + \text{sign}(a_1)\|a\|_2 e_1$ is our answer.

Applying Householder Transforms

- Don't compute Hx explicitly, that costs $3n^2$ flops.
- Instead use the formula given previously,

$$Hx = x - \left(2 \frac{v^T x}{v^T v} \right) v$$

which costs $4n$ flops (if you pre-compute $v^{\mathrm{T}} v$ or pre-normalize $v^{\mathrm{T}} v=2$).

- Typically, when using Householder transformations, you never compute the matrix H; it's only used in derivation and analysis.

QR Decomposition

- Householder transformations are a good way to zero out subdiagonal elements of a matrix.
- A is decomposed:

$$Q^T A = \begin{bmatrix} R \\ 0 \end{bmatrix} \quad \text{or} \quad QQ^T A = A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}$$

- where $Q^{\mathrm{T}}=H_n \ldots H_2 H_1$ is the orthogonal product of Householders and R is upper triangular.
- Overdetermined system $Ax=b$ is transformed into the easy-to-solve

$$\begin{bmatrix} R \\ 0 \end{bmatrix} x = Q^T b$$