

1. Write down true or false.

(3 points each)

- a) The formula  $y_{n+1} = y_n + 0.5 * h * (f_{n+1} + f_n)$  is an implicit formula for solving ordinary differential equations **T**
- b) The package ode23s performs worse in integrating a stiff ordinary differential equation than ode45 **F**
- c) Adaptive numerical integration routines use fewer points where the function is changing rapidly **F**
- d) The FFT allows the calculation of the Fourier transform of a  $N$  vector for a cost of  $O(N^2)$  operations. **F**

2. Given the system of ordinary differential equations

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u - 2 \cos v \\ u + tv^2 \end{bmatrix}$$

and the initial condition  $u(0) = 1$   $v(0) = \pi$

- a) write down the symbolic expression for the Jacobian matrix that determines stability for any value of  $t$  **10 points**
- b) Write down the numerical value of this matrix using the values given for  $t=0$ . **4 points**
- c) Determine the eigenvalues and eigenvectors of this matrix **10 points**

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 2u - 2 \cos v \\ u + tv^2 \end{bmatrix}, \text{ So the Jacobian is } \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 2 & 2 \sin v \\ 1 & 2tv \end{bmatrix}$$

Substituting the initial condition values ( $u = 1, t = 0, v = \pi/2$ ), the Jacobian matrix is  $\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$ .

This has eigenvalues given by the solution of

$$-\lambda(2 - \lambda) - 2 = 0 \quad \text{or} \quad \lambda^2 - 2\lambda - 2 = 0 \quad \text{or} \quad \lambda = (2 \pm \sqrt{4 + 8})/2 \quad \text{or} \quad \lambda = (1 \pm \sqrt{3})$$

Inserting the roots we find that the matrix respectively becomes

$$\begin{bmatrix} 1 - \sqrt{3} & 2 \\ 1 & -1 - \sqrt{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 + \sqrt{3} & 2 \\ 1 & -1 + \sqrt{3} \end{bmatrix}$$

Multiplying the second row in the first case by  $1 - \sqrt{3}$  and in the second case by  $1 + \sqrt{3}$  we get

$$\begin{bmatrix} 1 - \sqrt{3} & 2 \\ 1 - \sqrt{3} & (-1 - \sqrt{3})(1 - \sqrt{3}) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 + \sqrt{3} & 2 \\ 1 + \sqrt{3} & (-1 + \sqrt{3})(1 + \sqrt{3}) \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 1 - \sqrt{3} & 2 \\ 1 - \sqrt{3} & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 + \sqrt{3} & 2 \\ 1 + \sqrt{3} & 2 \end{bmatrix}$$

As expected the eigenvalues make the matrix degenerate, and the eigenvector is a vector orthogonal to the degenerate vector

$$\begin{bmatrix} -2 \\ 1 - \sqrt{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 1 + \sqrt{3} \end{bmatrix} \quad \text{can be taken as the corresponding eigen vectors}$$

d) Are solutions of this ode stable at  $t=0$ ?

5 points

Since one of the eigenvalues is positive we will have a growing solution and so the ode is not stable at  $t=0$

3) Suppose we have 25 data points  $(x_i, y_i)$  with  $x_1 < x_2 < \dots < x_n$ .

Write down a formula that approximates

$$Q = \int_{x_1}^{x_n} y(x) dx$$

using the values of  $(x_i, y_i)$  given.

20 points

Any formula is acceptable. The trapezoidal rule would have been

$$Q = 0.5(x_2 - x_1)(y_1 + y_2) + 0.5(x_3 - x_2)(y_2 + y_3) + \dots + 0.5(x_{25} - x_{24})(y_{25} + y_{24})$$

In fact since I did not say these points are equidistant, the other rules would have been a bit more complex to write

What is the order of approximation of the formula you have written?

5 points

As discussed in class the error in the composite trapezoid rule is proportional to the domain size (which is  $[x_{25}-x_1]$ ), the local interval squared ( $h^2$ ), and the second derivative. The trapezoidal rule is a first order method (or another acceptable way to say it would have been that it is exact for linear functions)

4) Provide brief answers

a. Given a  $m \times n$  matrix  $A$ . Write down the following decompositions. In each case mention the restrictions on  $m, n$ ; the types of matrices that form the decomposition (orthogonal, diagonal, triangular), the dimensions of the resulting matrices, and where such a decomposition is used in applications (8 points each)

i. QR decomposition

ii. Singular Value Decomposition

iii. Eigenvalue decomposition

b. Write down the QR algorithm for computing eigenvalues and eigenvectors of a matrix  
10 points

$A_0 = A$

for  $k=1, 2, \dots$

$$Q_k R_k = A_k$$

$$A_{k+1} = R_k Q_k$$

end

$A_{k+1}$  tends to an upper triangular matrix with the same eigenvalues as  $A$ . These eigenvalues lie along the main diagonal of  $A_{k+1}$ . Once the eigenvalues are known, the eigenvector can be solved by solving the homogeneous linear systems.