Polynomial Interpolation

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Interpolation

• Given a function at $N$ points, find its value at other point(s)
  – “within” the points
    • Interpolation, Imputation
  – “outside” the points
    • Extrapolation, Forecasting

• How do we extend the value from known points to unknown points?
  – Have to have prior knowledge about the function
  – Can do what is convenient

• Occam’s razor
  – Parsimony: “One should not increase, beyond what is necessary, the number of entities required to explain anything”
  – Key in all scientific modeling
In interpolation sampled values of a function are available

We need to extend these values to points where they are not available

Can assume that the function is expanded over a basis of functions which span the functional space

- Polynomials are a basis -- Fourier series are another
- Many others

Knowing coefficients specifies the function

Most common basis are power series/polynomials

- Here $x_*$ is a point about which we expand in series
- $a_m$ are coefficients

Polynomial interpolation is often natural

- Part of theory of Taylor series, solution of differential equations via power series, in computing integrals
- A theorem guaranteeing that a polynomial can represent a function is available
Interpolation

• Weierstrass theorem

**Theorem:** (Weierstrass) For all $f \in C[a, b]$, for all $\epsilon > 0$, there exists a degree $n$ and a polynomial $p_n$ such that $\|f - p\|_\infty < \epsilon$.

• Provides a guarantee that polynomials can interpolate any function
• On the other hand does not tell us how to choose the polynomial
• Also does not guarantee that the polynomial will actually “interpolate” the function … only that it will be within $\epsilon$.
• Does not tell what the degree of the polynomial is
Taylor Series

- Let $f(y)$ be a real function, $f(y) \in C^n [x_*, x_*+r)$
  - $n^{th}$ derivative $f^n$ exists for $x_* \leq y < x_*+r$

\[
f(y) = f(x_*) + f'(x_*)(y-x_*) + \frac{1}{2!}f''(x_*)(y-x_*)^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(x_*)(y-x_*)^{n-1} + \text{Residual}_n(y).
\]

- Residual determines accuracy
- Two evaluations of remainder
  - Cauchy evaluation
  - Lagrange evaluation

\[
|\text{Residual}_n(y)| \leq \frac{|y-x_*|^n}{n!} \sup_{x_* \leq x < x_*+r} |f^{(n)}(y)|.
\]

\[
\text{Residual}_n(y) = \int_{x_*}^{y} dx \int_{x_*}^{x} dx \ldots \int_{x_*}^{x} f^{(n)}(X)dx = \frac{1}{n!}f^{(n)}(X)(y-x_*)^n,
\]

$X \in (x_*, x_*+r)$. 
Polynomial Facts

- A polynomial of degree $k$ has at most $k$ distinct zeroes, unless it is identically zero.
- Sum of two polynomials of degree $k$ is another polynomial of degree at most $k$.
- Polynomials can be expressed in many ways.

For example $(x - 2)(x - 5) = x^2 - 2x - 5(x - 2) = x^2 - 7x + 10$.

We have used three different sets of basis functions in this example:

1. $(x - 2)(x - 5)$, $x - 1$, and 1.
2. $x^2$, $x$, and $x - 2$.
3. $x^2$, $x$, and 1.
   - Degree of basis functions is 2, 1 and 0 …
   - Basis 3 is the power basis or monomial basis
   - Any basis can be used … often “orthogonal polynomials” are used
Uniqueness of interpolant

- We know that the polynomial exists
- Suppose that there are two different polynomials that can interpolate the data
- Let them be \( p_{n-1} \) and \( q_{n-1} \).
- So we have
  \[
  p_{n-1}(x_i) = y_i, \quad i = 1, \ldots, n \\
  q_{n-1}(x_i) = y_i, \quad i = 1, \ldots, n
  \]
- So \( p_{n-1}(x_i) - q_{n-1}(x_i) = 0, \quad i = 1, \ldots, n \)
- \( p_{n-1} - q_{n-1} \) is the difference of two polynomials of degree \( n-1 \).
- It has \( n \) zeroes.
- Recall polynomial of degree \( k \) has at most \( k \) zeroes, or is the zero polynomial.
- Here we have more zeroes than degree \( n \) … so it is the zero polynomial.
- So interpolant is unique.
Interpolating polynomials in power form

- Given $n$ values of the function $y_i$ at points $x_i$
- Fit a polynomial $P(x)$ that interpolates data at these points
- If we have $n$ points to interpolate, then a polynomial of degree $n-1$ is
  \[ P(x) = c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-1} x + c_n \]
- We can write the condition that it interpolate as a linear system
  \[
  \begin{pmatrix}
  x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\
  x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\
  \vdots & \vdots & \cdots & \vdots & \vdots \\
  x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1
  \end{pmatrix}
  \begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
  \end{pmatrix}
  =
  \begin{pmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_n
  \end{pmatrix}
  \]