Computational Methods
CMSC/AMSC/MAPL 460

Linear Systems, LU Decomposition,

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Matrix norms

- Can be defined using corresponding vector norms
  - Two norm
  - One norm
  - Infinity norm
- Two norm is hard to define ... need to find maximum singular value
  - related to idea that matrix acting on unit sphere converts it in to an ellipsoid
- Frobenius norm is defined just using matrix elements
Condition Number of a Matrix

A measure of how close a matrix is to singular

\[
\text{cond}(A) = \kappa(A) = \|A\| \cdot \|A^{-1}\| = \frac{\text{maximum stretch}}{\text{maximum shrink}} = \frac{\max |\lambda_i|}{\min |\lambda_i|}
\]

- \text{cond}(I) = 1
- \text{cond(singular matrix)} = \infty
Solving Linear Systems

- One idea compute inverse
- Not usually a good idea
  - (unless inverse is computable easily and accurately using some matrix property)
- Leads to increased errors, and is more expensive usually

\[ Ax = b \]

\[ 7x = 21 \]

\[ x = \frac{21}{7} = 3 \]

\[ x = 7^{-1} \times 21 \]

\[ = .142857 \times 21 = 2.99997 \]
Representing linear systems as matrix-vector equations

\[
\begin{align*}
10x_1 - 7x_2 &= 7 \\
-3x_1 + 2x_2 + 6x_3 &= 4 \\
5x_1 - x_2 + 5x_3 &= 6
\end{align*}
\]

• Represent it as a matrix-vector equation (linear system)
• We will apply the familiar elimination technique, and then see its matrix equivalent

\[
\begin{pmatrix}
10 & -7 & 0 \\
-3 & 2 & 6 \\
5 & -1 & 5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
7 \\
4 \\
6
\end{pmatrix}
\]
Cramer’s Rule

- Familiar algorithm often taught in school years
- Get determinant of the matrix
  - Determinant defined by recursion
  - Complexity of determinant determination
  - Factorial and polynomial complexity
- Get determinant of the matrix with column corresponding to variable replaced with the right hand side

\[
x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}
\]
Cramer’s Rule (contd.)

• In 3D

\[
(x, y, z) = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D}, \quad \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D}, \quad \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}.
\]

• In higher dimensions it generalizes similarly

• Home reading:
  – Why does Cramer’s rule work?
  – (From “Cramer’s rule article Eric W. Weisstein  math encyclopedia … available online)
Gaussian Elimination

- Zero elements of first column below 1\(^{st}\) row
  - multiplying 1\(^{st}\) row by 0.3 and add to 2\(^{nd}\) row
  - multiplying 1\(^{st}\) row by -0.5 and add to 3\(^{rd}\) row
  - Results in

\[
\begin{pmatrix}
10 & -7 & 0 \\
-3 & 2 & 6 \\
5 & -1 & 5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
7 \\
4 \\
6
\end{pmatrix}
\]

- Zero elements of first column below 2\(^{nd}\) row
  - Swap rows
  - Multiply 2\(^{nd}\) row by 0.04 and add to 3\(^{rd}\)

\[
\begin{pmatrix}
10 & -7 & 0 \\
0 & 2.5 & 5 \\
0 & -0.1 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
7 \\
2.5 \\
6.1
\end{pmatrix}
\]

\[
\begin{pmatrix}
10 & -7 & 0 \\
0 & 2.5 & 5 \\
0 & 0 & 6.2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
7 \\
2.5 \\
6.2
\end{pmatrix}
\]
Solution

• Start from last equation which can be solved by division
• Next substitute in the previous line and continue
• This describes the way to do the algorithm by hand
• How to represent it using matrices?

• Also, how do we solve another system that has the same matrix?
  – Upper triangular matrix we end up with will be the same, but the sequence of operations on the r.h.s needs to be repeated

\[
\begin{align*}
6.2x_3 &= 6.2 \\
2.5x_2 + (5)(1) &= 2.5, \\
10x_1 + (-7)(-1) &= 7
\end{align*}
\]

\[x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\]
Gaussian Elimination: LU Matrix decomposition

- It turns out that Gaussian elimination corresponds to a particular matrix decomposition …
  - Product of permutation, lower triangular and upper triangular matrices

- What is a permutation matrix?
  - It rearranges a system of equations and changes the order.
  - Multiplying by it swaps the order of rows in a matrix
  - Essentially a rearrangement of the identity
  - Nice property: transpose is its inverse: $PP^T = I$

\[
P = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

\[
P x = b
\]

\[
x = P^T b
\]
LU Decomposition

- What is an upper triangular matrix?
  - Elements below diagonal are zero

- Lower triangular matrix
  - Elements above diagonal are zero

- Unit lower triangular matrix
  - Elements along diagonal are one

- Upper triangular part of Gauss Elimination is clear ...
  - Final matrix we end up with

- What about lower triangular and permutation?

\[ U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} \]

\[ L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ 4 & 6 & 7 & 1 \end{pmatrix} \]
• Identify the elements of \( L \) and \( P \)?
• \( L \) has the multipliers we used in the elimination steps
• \( P \) has a record of the row swaps we did to avoid dividing by small numbers
• In fact we can write each step of Gaussian elimination in matrix form

\[
L = \begin{pmatrix}
1 & 0 & 0 \\
0.5 & 1 & 0 \\
-0.3 & -0.04 & 1 \\
\end{pmatrix} \quad U = \begin{pmatrix}
10 & -7 & 0 \\
0 & 2.5 & 5 \\
0 & 0 & 6.2 \\
\end{pmatrix} \quad P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

\[
U = M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 A \\
L_1 L_2 \cdots L_{n-1} U = P_{n-1} \cdots P_2 P_1 A
\]
\[
A = \begin{pmatrix}
10 & -7 & 0 \\
-3 & 2 & 6 \\
5 & -1 & 5
\end{pmatrix}
\]

\[
L = L_1 L_2 \cdots L_{n-1}
\]

\[
P = P_{n-1} \cdots P_2 P_1
\]

the matrices defined during the elimination are

\[
P_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad M_1 = \begin{pmatrix}
1 & 0 & 0 \\
0.3 & 1 & 0 \\
-0.5 & 0 & 1
\end{pmatrix},
\]

\[
P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0.04 & 1
\end{pmatrix},
\]
$$A = \begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.04 & 1 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.04 & 1 \end{pmatrix},$$
Solving a system with the LU decomposition

\[ Ax = b \]
\[ LU = PA \]
\[ P^T LUx = b \]
\[ L[Ux] = Pb \]
Solve \( Ly = Pb \)
Then \( Ux = y \)
Solving a triangular system

\[ x = \text{zeros}(n,1); \]
\[ \text{for } k = n:-1:1 \]
\[ x(k) = b(k)/U(k,k); \]
\[ i = (1:k-1)'; \]
\[ b(i) = b(i) - x(k)*U(i,k); \]
\[ \text{end} \]

\[ x = \text{zeros}(n,1); \]
\[ \text{for } k = n:-1:1 \]
\[ j = k+1:n; \]
\[ x(k) = (b(k) - U(k,j)*x(j))/U(k,k); \]
\[ \text{end} \]