Computational Methods

CMSC/AMSC/MAPL 460

Representing numbers in floating point
Error Analysis

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Class Outline

- Recap of floating point representation
- Matlab demos
- Error analysis
- Forward error analysis
- Backward error analysis

Floating point representation

- Represent real numbers using a finite number of bits
- Keywords
  - Mantissa, Exponent, Sign
  - Precision
  - Machine Epsilon
  - Overflow, Underflow
- Special Numbers
  - Zero
  - Inf
  - NaN
  - Underflow

Can be written...

0	00000000000	0000000000000......000000000000000
sign	exponent	mantissa (significand)

$$(-1)^{S} * 2^{E} * 1.f$$

	E+1023 == 0	0 < E+1023 < 2047	E+1023 == 2047
f==0	0	Powers of Two	∞
f~=0	Non-normalized typically underflow	Floating point Numbers	Not A Number

Exponent

- IEEE-Double: stored as binary number+1023
  - Also known as **biased** exponent
- Occupies 11 bits
  - Decimals values range from 0 to 2^{11} -1 = 2047
  - Exponent values used to represent numbers range from -1022 to 1023 (1 to 2046)
  - Values -1023 and 1024 are special
- IEEE-Single: stored as binary number+127
- Occupies 8 bits
  - Decimals values range from 0 to 2^{8} -1 = 255
  - Exponent values used to represent numbers range from -126 to 127 (1 to 254)
  - Values -127 and 128 are special

Sign and Mantissa

- Sign is 0 (positive) or 1 (negative)
- -1^s
- Mantissa: Has the form 1+0.f
- Double precision
  - f is a 52 bit number
  - Abs min is 0
  - max is $½+1/2^2+1/2^3+\ldots+1/2^{52}$
  - Geometric series $½(1-(1/2)^{52})/(1-1/2)$
    $= (1-(1/2)^{52}) = 1-2.22044604925031e-016$
- single precision
  - 23 bit number in
  - Abs min 0, max is $½+1/2^2+1/2^3+\ldots+1/2^{23}$
  - Geometric series $1-(1/2)^{23}=0.99999988079071$

Precision and Machine ε

- Any two numbers for a given value of the exponent are separated by (2^-52)*2^(e-1023) = 2^(e-1023-52)
- For each value of e the separation is uniform
- *machine epsilon*: eps *is the distance from* 1 *to the next larger floating-point number.*
- Floatgui Matlab code

Special numbers

- Zero
  - Since numbers are written as $(-1)^s\,(1+f)*2^{(e-1023)}$ we cannot have zero
  - So zero must be specially coded
  - Choose the lowest value: e=0 and f=0
  - (without this understanding the number would be $2^{-1023} = 1.1125369292536 \times 10^{-308}$)
- Infinity
  - Corresponds to f=0 and e=2047
  - Without this understanding would be $1.79769313486232\times 10^{309}$
- Undefined numbers
  - If any computation tries to produce a value that is undefined even in the real number system, the result is an "exception" known as Not-a-Number, or NaN.
  - Examples include 0/0 and Inf-Inf.
  - NaN is represented by taking e = 1024 and f nonzero.
  - "Floating point exception"

Other exceptions

- Overflow: calculation yields number larger than realmax
- Underflow: calculations yields number smaller than realmin

Number	Binary	Decimal
eps	2^{-52}	2.2204e-16
realmin	2^{-1022}	2.2251e-308
realmax	$(2\text{-eps})*2^{1023}$	1.7977e+308

Some numbers cannot be exactly represented

$$\frac{1}{10} = \frac{1}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \frac{0}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \frac{0}{2^{10}} + \frac{0}{2^{11}} + \frac{1}{2^{12}} + \ldots$$

$$t = (1 + \frac{9}{16} + \frac{9}{16^2} + \frac{9}{16^3} + \ldots + \frac{9}{16^{12}} + \frac{10}{16^{13}}) \cdot 2^{-4}$$

$$t_1 < 1/10 < t_2$$

Where

$$t_1 = 2^{-4} \cdot (1 + \frac{9}{16} + \frac{9}{16^2} + \frac{9}{16^3} + \ldots + \frac{9}{16^{12}} + \frac{9}{16^{13}})$$

$$t_2 = 2^{-4} \cdot (1 + \frac{9}{16} + \frac{9}{16^2} + \frac{9}{16^3} + \ldots + \frac{9}{16^{12}} + \frac{10}{16^{13}})$$

It turns out that $1/10$ is closer to t_2 than to t_1, so t is equal to t_2. In other words,

Effects of floating point errors

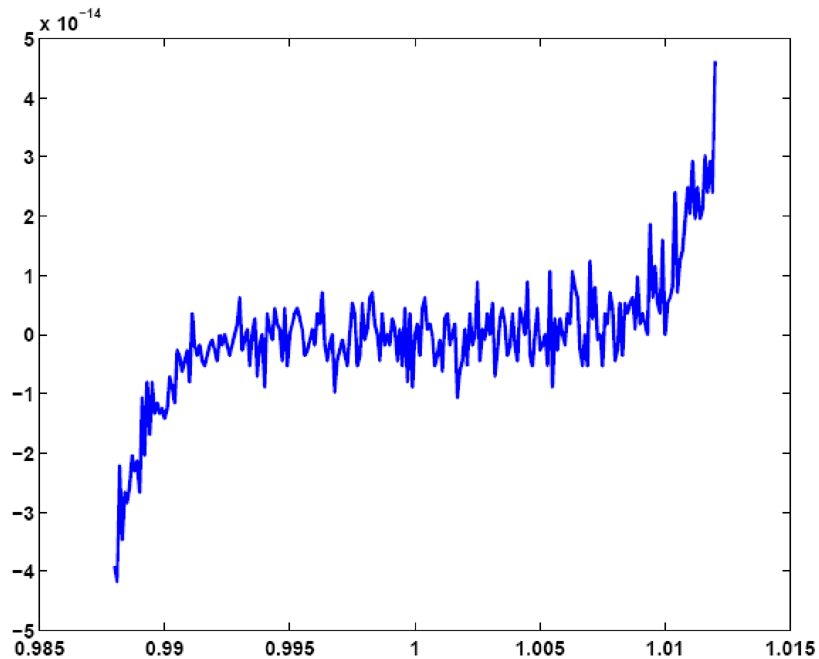
- Singular equations will only be nearly singular
- Severe cancellation errors can occur

$$\begin{aligned} 17x_1 + 5x_2 &= 22 \\ 1.7x_1 + 0.5x_2 &= 2.2 \end{aligned}$$

```
A = [17 5; 1.7 0.5]
b = [22; 2.2]
x = A\b
```

```
x = 0.988:.0001:1.012;
y = x.^7-7*x.^6+21*x.^5-35*x.^4+35*x.^3-21*x.^2+7*x-1;
plot(x,y)
```

produce




```
x =
  -1.0588
   8.0000
```

Measuring error

- **Absolute error** in c as an approximation to x:

$$|x - c|$$

- **Relative error** in c as an approximation to nonzero x:

$$|(x - c)/x|$$

Errors can be magnified

- Errors can be magnified during computation.
- Let us assume numbers are known to 0.05% accuracy
- Example: 2.003×10^0 and 2.000×10^0
  - both known to within $\pm .001$
- Perform a subtraction. Result of subtraction:

$$0.003 \times 10^0$$

- but true answer could be as small as 2.002 - 2.001 = 0.001,
- or as large as 2.004 - 1.999 = 0.005!
- Absolute error of 0.002
- Relative error of 200% !
- **Adding or subtracting causes the bounds on absolute errors to be added**

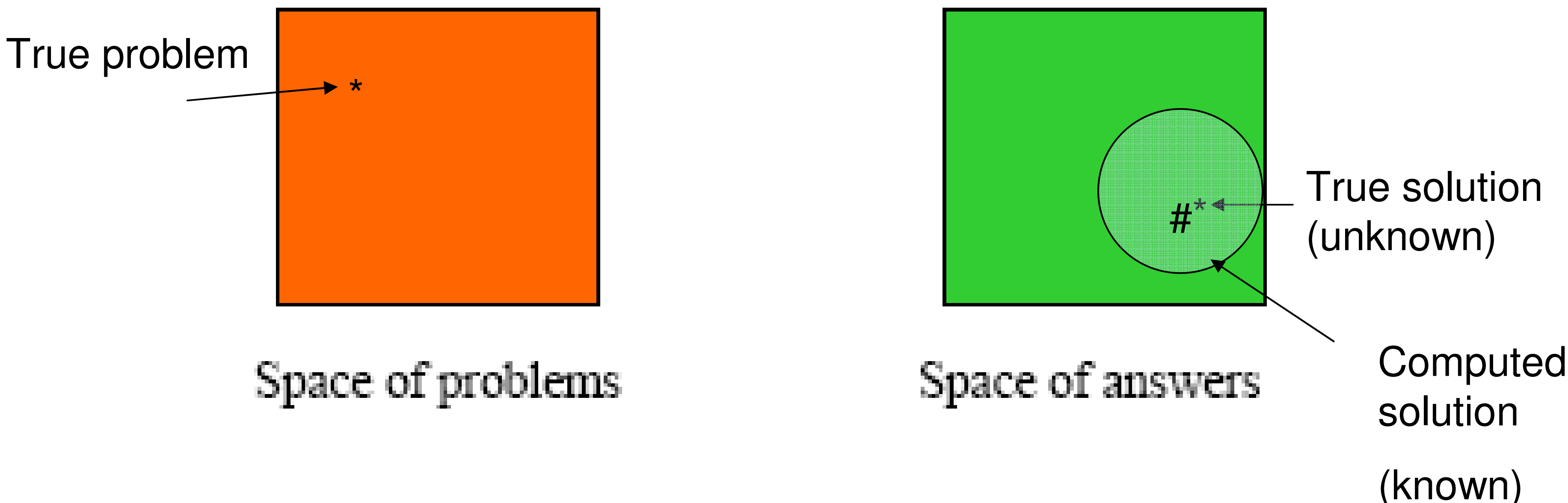
Error effect on multiplication/division

- Let x and y be true values
- Let $X=x(1+r)$ and $Y=y(1+s)$ be the known approximations
- Relative errors are r and s
- What is the errors in multiplying the numbers?
- XY=xy(1+r)(1+s)
- Absolute error $=|xy(1-rs-r-s-1)|= (rs+r+s)xy$
- Relative error = $|(xy-XY)/xy|$

  $= |rs+r+s| <= |r|+|s|+|rs|$
- If r and s are small we can ignore $|rs|$
- Multiplying/dividing causes relative error bounds to add

Error Analysis

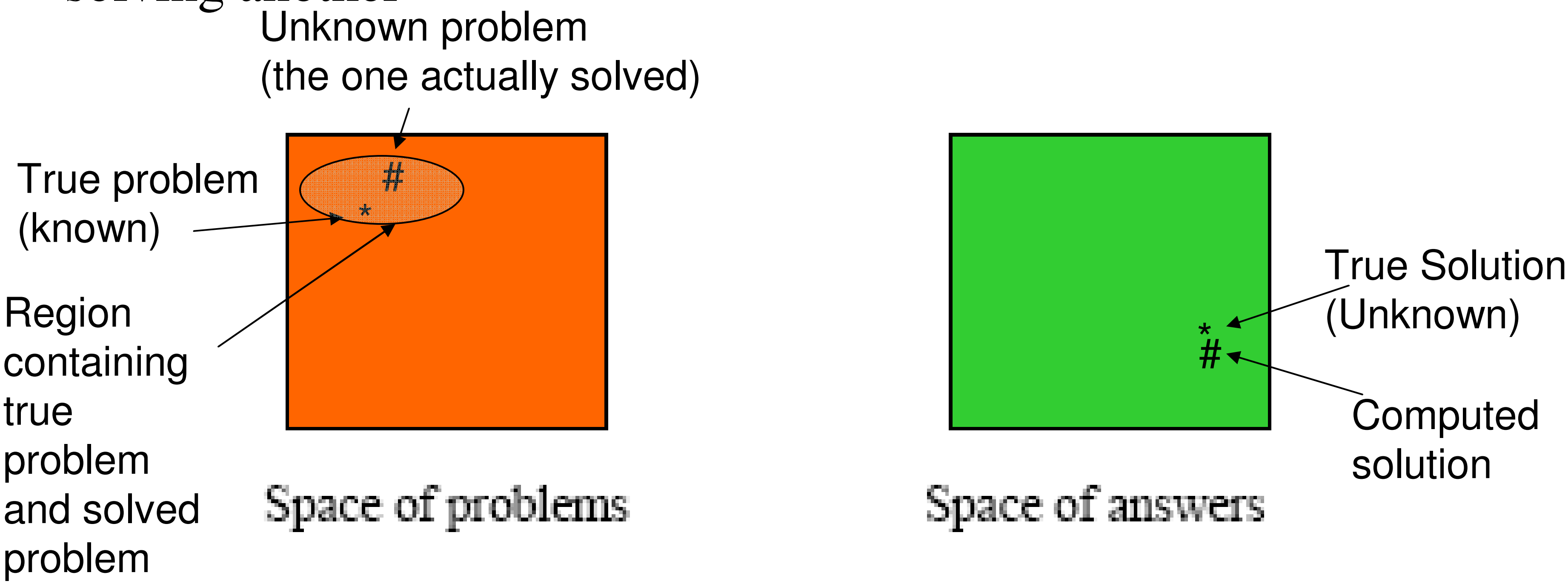
- Forward and Backward error analysis
- Forward error analysis
  - Assume that the problem we are solving is exactly specified
  - Produce an approximate answer using the algorithm considered



  - Goal of forward error analysis produce region guaranteed to contain true soln.
  - Report region and computed solution

Backward error analysis

- We know that our problem specification itself has error ("error in initial data")
- So while we think we are solving one problem we are actually solving another



- Given an answer, determine how close the problem actually solved is to the given problem.
- Report solution and input region

Well posed problems

- Hadamard postulated that for a problem to be “well posed”
  1. *Solution must exist*
  2. *It must be unique*
  3. *Small changes to input data should cause small changes to solution*
- Essentially this means the regions in the problem space and solution space must be small