Heat equation

$$\frac{u(\vec{x}, t+\delta) - u(\vec{x}, t)}{\delta} = \triangle_h u(\vec{x})$$

$$u(\vec{x}, 0) = u_0(\vec{x})$$

$$u(\vec{x}, t+\delta) = u(\vec{x}, t) + \delta \triangle_h u(\vec{x}, t)$$

Wave equation

$$\frac{u(\vec{x}, t+\delta) - 2u(\vec{x}, t) + u(\vec{x}, t-\delta)}{\delta^2} = \triangle_h u(\vec{x}, t)$$

$$\frac{\partial u}{\partial t}(\vec{x}, 0) = 0$$

$u(\vec{x}, 0) = u_0(\vec{x})$, and $u(\vec{x}, \delta) = u_0(\vec{x})$

$$u(\vec{x}, t+\delta) = 2u(\vec{x}, t) - u(\vec{x}, t-\delta) + \delta^2 \triangle_h u(\vec{x}, t)$$

Matrix representation in 1-D

- 1-D Poisson

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

With u represented as a vector and $h^2\triangle_h$ as a matrix A, the Poisson problem becomes

$$Au = b$$

where b is a vector (the same size as u) containing the values of $h^2 f(x)$ at the interior mesh points. The first and last components of b would also include any nonzero boundary values.

```
u = A\b
```

In Matlab this takes advantage of the sparse structure of A

Matrix representation in 2D

- Need to map 2D domain in to 1-D element
- Assume a L shaped region

```
L =

     0     0     0     0     0     0     0     0     0     0     0
     0     1     5     9    13    17    21    30    39    48     0
     0     2     6    10    14    18    22    31    40    49     0
     0     3     7    11    15    19    23    32    41    50     0
     0     4     8    12    16    20    24    33    42    51     0
     0     0     0     0     0     0    25    34    43    52     0
     0     0     0     0     0     0    26    35    44    53     0
     0     0     0     0     0     0    27    36    45    54     0
     0     0     0     0     0     0    28    37    46    55     0
     0     0     0     0     0     0    29    38    47    56     0
     0     0     0     0     0     0     0     0     0     0     0
```

- Boundary conditions on 1-4, 5,8,9,12,13,16, 17, 20, 21, 29,30,38,39,47, 48-56

- Write the operator as Au=f
- Consider node 43

33	42	51
34	43	52
35	44	53

$$h^2 \triangle_h u(43) = u(34) + u(42) + u(44) + u(52) - 4u(43)$$

- So the elements of the matrix a for the row corresponding to 43 are

$$a_{43,34} = a_{43,42} = a_{43,44} = a_{43,52} = 1, \text{ and } a_{43,43} = -4$$

- All other entries are zero
- Each row for an internal node can have at most 5 non-zero entries

Stability for time-dependent equations

- Can study stability as we did for ODEs
- Gives a condition relating timestep size and mesh size
- Heat equation

$$u^{(k+1)} = u^{(k)} + \sigma A u^{(k)}$$

$$\sigma = \frac{\delta}{h^2}$$

$$u^{(k+1)} = M u^{(k)}$$

$$M = I + \sigma A$$

- For stability we require in one dimension
- So mesh size and time step cannot be chosen independently

$$\sigma \le \frac{1}{2}$$

In two dimensions

$$\sigma \le \frac{1}{4}$$

Wave equation stability criteria

$$u^{(k+1)} = 2u^{(k)} - u^{(k-1)} + \sigma A u^{(k)}$$

$$\sigma = \frac{\delta^2}{h^2}$$

For the wave equation in one dimension

$$\sigma \leq 1$$

In two dimensions

$$\sigma \leq \frac{1}{2}$$

Wave equation

- Separation of variables

$$\frac{\partial^2 u}{\partial t^2} = \triangle u$$

$$u(\vec{x}, t) = \cos(\sqrt{\lambda}\, t)\ v(\vec{x})$$

- Gives the "Helmholtz" equation for *v*

$$\triangle v + \lambda v = 0$$

- Solving for *v*
- So overall solutions is

In one dimension

$$v_k(x) = \sin(kx)$$

$$v_k(\pi) = 0$$

k must be an integer

$$\lambda_k = k^2.$$

$$\begin{aligned} u(x,t) &= \sum_k a_k \cos(kt)\sin(kx) \\ &= \sum_k a_k \cos(\sqrt{\lambda_k}\, t)\ v_k(x) \end{aligned}$$