Computational Methods
CMSC/AMSC/MAPL 460

Partial Differential Equations

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Partial Differential equations

• Given a differential equation involving a function of a single variable we get an ordinary differential equation
• Partial differential equations involve the solution of functions of more than one variable
• Typically involve “boundary conditions” on spatial domains and “initial conditions” on time
• Several methods to numerically solve them
• Here we will look at the simplest
  – Finite difference methods on “regular regions”
Initial and Boundary value problems

- Initial value problems are the kind we have seen so far in the ODE part of the course
- Involve finding $y(t)$, given an ordinary differential equation and initial conditions $y(t_0)$
- We may also have boundary value problems
- Given a differential equation and conditions at two values of the independent variable
  - $y(t_0) = y_0$
  - $y(t_1) = y_1$
Laplace’s Equation and Poisson’s Equation

• Laplacian

In one space dimension
\[ \Delta = \frac{\partial^2}{\partial x^2} \]

In two space dimensions
\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

• Poisson’s equation

\[ \Delta u = f(\bar{x}) \]

• Laplace’s equation

• \( f=0 \)

• “Elliptical Equations”
Heat equation

\[
\frac{\partial u}{\partial t} = \triangle u - f(\vec{x})
\]

\[
u(\vec{x}, 0) = u_0(\vec{x})
\]

- Parabolic equation
- Boundary conditions as for Laplace, plus initial conditions at time \( t=0 \)
Wave equation

\[ \frac{\partial^2 u}{\partial t^2} = \Delta u \]

\[ u(\vec{x}, 0) = u_0(\vec{x}) \]

\[ \frac{\partial u}{\partial t}(\vec{x}, 0) = 0 \]

- Hyperbolic equation
- Boundary conditions on space, plus two initial conditions
Finite difference methods

• Approximate the action of operator on function values

In one dimension

\[ a \leq x \leq b \]

\[ h = \frac{(b - a)}{(m + 1)} \]

\[ x_i = a + i h, \quad i = 0, \ldots, m + 1 \]

\[ \triangle_h u(x) = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} \]
Two dimensions

\[(x_i, y_j) = (ih, jh)\]

\[\triangle_h u(x, y) = \frac{u(x + h, y) - 2u(x, y) + u(x - h, y)}{h^2} + \frac{u(x, y + h) - 2u(x, y) + u(x, y - h)}{h^2}\]
Laplacian Stencil

\[ P = (x, y) \]
\[ N = (x, y + h) \]
\[ E = (x + h, y) \]
\[ S = (x, y - h) \]
\[ W = (x - h, y) \]

\[ \triangle_h u(P) = \frac{u(N) + u(W) + u(E) + u(S) - 4u(P)}{h^2} \]