Computational Methods
CMSC/AMSC/MAPL 460

Fourier transform

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Several slides from Prof. Healy’s course at UMD
Fourier Methods

• Fourier analysis ("harmonic analysis") a key field of math
• Has many applications and has enabled many technologies.
• Basic idea: Use Fourier representation to represent functions
• Has fast algorithms to manipulate them (the fast Fourier Transform)
• Requires a complete course *ENEE425: Digital Signal Processing* or *MATH464: Introduction to Fourier Analysis*
Basic idea

- Function spaces can have many different types of bases
- We have already met monomials and other polynomial basis functions
- Fourier introduced another set of basis functions: the Fourier series
- These basis functions are particularly good for describing things that repeat with time
Fourier’s Representation

\[ F(t) = A_0/2 + A_1 \cos(t) + A_2 \cos(2t) + A_3 \cos(3t) + \ldots \]
\[ + B_1 \sin(t) + B_2 \sin(2t) + B_3 \sin(3t) + \ldots \]

If the sequence \( \{A_1, A_2, \ldots\} \) and \( \{B_1, B_2, \ldots\} \) goes to 0 fast enough, these sums will converge to \( F(t) \) at each value of \( t \).

This defines a new function, which must be a \textit{periodic function}. (Period \( 2 \pi \))

Fourier’s claim: ANY periodic function \( f(t) \) can be written this way
Fourier’s Representation

$1 \sin(t) + \ldots$
Fourier’s Representation

\[ \sin(t) - \frac{1}{2} \sin(2t) + \ldots \]
Fourier’s Representation

\[ 1 \sin( t) - \frac{1}{2} \sin( 2 t) + \ldots \]
Fourier’s Representation

\[ 1 \sin(t) - \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \ldots \]
Fourier’s Representation

\[ 1 \sin(t) - \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \ldots \]
Fourier’s Representation

20’th degree Fourier expansion
Synthesize Music using Fourier Series!
http://www.phy.ntnu.edu.tw/ntnujava/viewtopic.php?t=33

- Similar to a music synthesizer
Fourier series for periodic extension of nonperiodic function

Represent $f(t) = t$ for $|t| < \pi$ and $2\pi$ periodic

Just a ramp function repeated
How do you get the Coefficients for a given $f$?

$$A_0/2 + A_1 \cos(t) + A_2 \cos(2t) + A_3 \cos(3t) + \ldots$$

$$+ B_1 \sin(t) + B_2 \sin(2t) + B_3 \sin(3t) + \ldots$$

Fourier’s claim:

- *ANY* periodic function $f(t)$ can be written this way (SYNTHESIS)
- The coefficients are uniquely determined by $f$: (ANALYSIS)

$$A_k = \frac{1}{2} \pi \int_{0}^{2\pi} f(t) \cos(kt) \, dt$$

$$B_k = \frac{1}{2} \pi \int_{0}^{2\pi} f(t) \sin(kt) \, dt$$
Fourier Analysis

- Def.: mathematical techniques for breaking up a signal into its components (sinusoids)
- Jean Baptiste Joseph Fourier (1768-1830)
- can represent any continuous periodic signal as a sum of sinusoidal waves
Fourier Analysis: match data with sinusoids

\[ S[k] = \int s(t) \cos(kt) \, dt \]
**Complex Notation**

For $f$, periodic with period $p$ combine sines and cosines using identity $\exp(i \theta) = \cos \theta + i \sin \theta$

**Fourier transform** $f(t) \rightarrow F[k]$

$$F[k] = \frac{1}{p} \int_{0}^{p} f(t) \cos(2\pi \frac{k t}{p}) \, dt$$

$$- \frac{i}{p} \int_{0}^{p} f(t) \sin(2\pi \frac{k t}{p}) \, dt$$

$$= \frac{1}{p} \int_{0}^{p} f(t) e^{-2\pi i \frac{k t}{p}} \, dt$$

**Inverse Fourier transform** $F[k] \rightarrow f(t)$

$$f(t) = \sum F[k] e^{2\pi i \frac{k t}{p}}$$
DFT and its inverse for periodic discrete data

\[
\Phi[k] = \sum_{n=0}^{N-1} \phi[n] \, e^{-2\pi i k n h / p} \quad p = N \, h
\]

This is automatically periodic in \( k \) with period \( N \)

Inverse is like Fourier series, but with only \( p \) terms
Two PERIODIC time versions of Fourier

\[ \int_0^p f(t) e^{-2\pi i k t/p} \, dt / p \]

\[ \sum_{k \in \mathbb{Z}} F[k] e^{2\pi i k t/p} \]

\[ f(t), \text{ period } p \]

\[ S_{p/N} \]

\[ \mathcal{F}[k], \text{ on } \mathbb{Z} \]

\[ F[k], \text{ on } \mathbb{Z} \]

\[ \sum_{k = 0}^{N-1} \Gamma[k] e^{-2\pi i m k/N} \]

\[ \sum_{k = 0}^{N-1} \Gamma[k] e^{2\pi i m k/N} \]

\[ \gamma[k], \text{ on } P_N \]

\[ \mathcal{F}[m], \text{ on } P_N \]

i.e. on \( \mathbb{Z} \), Period N

\[ \text{“time” domain} \]

\[ \text{“frequency” domain} \]
DFT: Discrete time periodic version of Fourier

"time" domain

\[
\gamma[k] = \sum_{k=0}^{N-1} \gamma[k] e^{-2\pi i k m/N} = \Gamma[m]
\]

\[
\Gamma[m] = \sum_{k=0}^{N-1} \Gamma[k] e^{2\pi i m k/N}
\]

\gamma[k], \text{ on } \mathcal{P}_N

i.e. on \mathbb{Z}, \text{ Period } N

"frequency" domain

\[
\Gamma[m] \text{ on } \mathcal{P}_N
\]

i.e. on \mathbb{Z}, \text{ Period } N