Computational Methods
CMSC/AMSC/MAPL 460

Fourier transform

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Several slides from Prof. Healy's course at UMD

Fourier Methods

- Fourier analysis ("harmonic analysis") a key field of math
- Has many applications and has enabled many technologies.
- Basic idea: Use Fourier representation to represent functions
- Has fast algorithms to manipulate them (the fast Fourier Transform)
- Requires a complete course *ENEE425: Digital Signal Processing* or *MATH464: Introduction to Fourier Analysis*

Basic idea

- Function spaces can have many different types of bases
- We have already met monomials and other polynomial basis functions
- Fourier introduced another set of basis functions : the Fourier series
- These basis functions are particularly good for describing things that repeat with time

Fourier's Representation

$$F(t) = A_0/2 + A_1 \operatorname{Cos}(t) + A_2 \operatorname{Cos}(2t) + A_3 \operatorname{Cos}(3t) + \ldots$$

$$+ B_1 \operatorname{Sin}(t) + B_2 \operatorname{Sin}(2t) + B_3 \operatorname{Sin}(3t) + \ldots$$

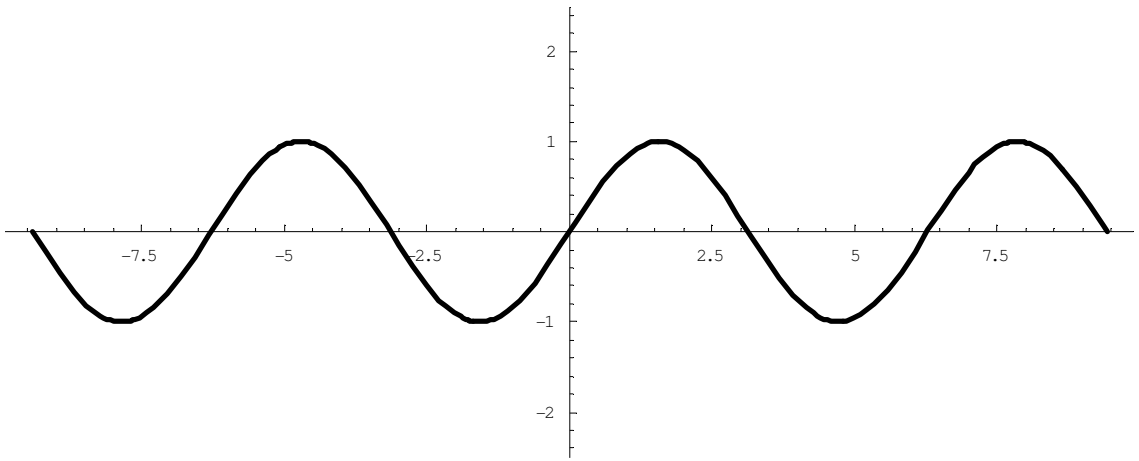
If the sequence $\{A_1, A_2, \ldots\}$ and $\{B_1, B_2, \ldots\}$ goes to 0 fast enough, these sums will converge to F(t) at each value of t .

This defines a new function, which must be a ***periodic function***. (Period 2π)

Fourier's claim: *ANY* periodic function *f(t)* can be written this way

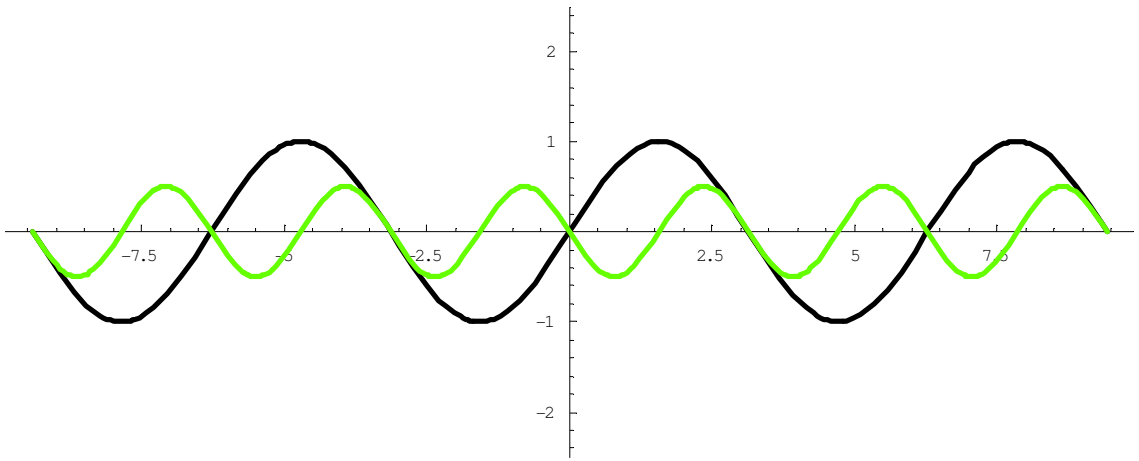
Fourier's Representation

1 Sin(t) + …



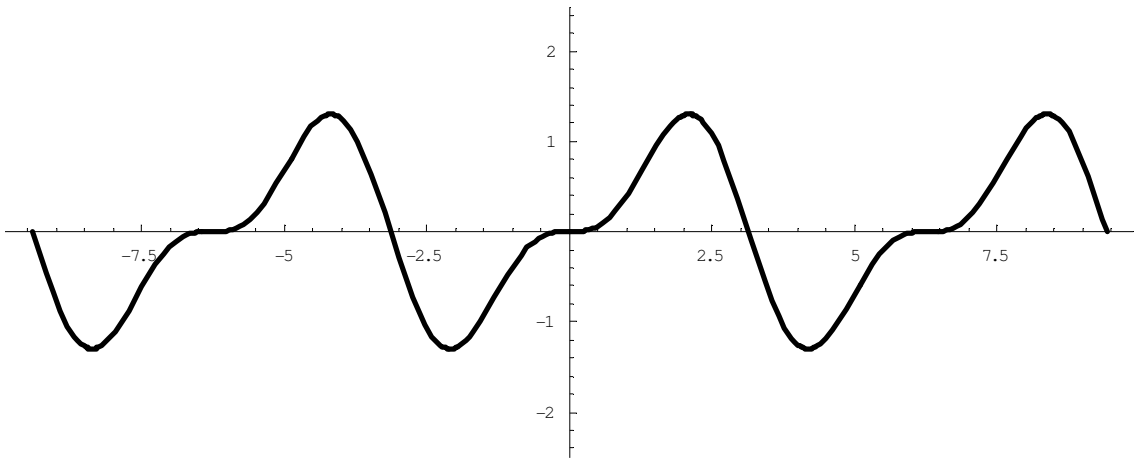
Fourier's Representation

Sin(t) - 1/2 Sin(2 t) + …



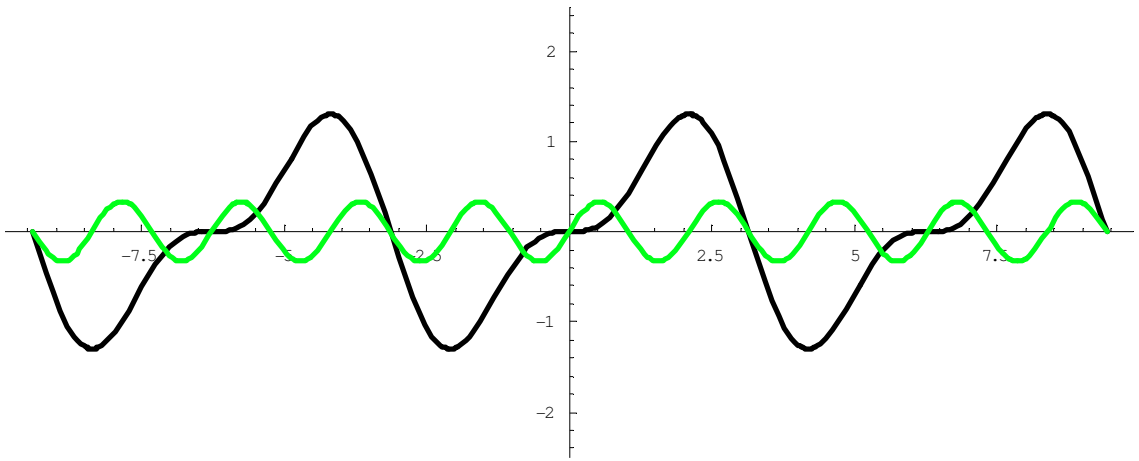
Fourier's Representation

1 Sin(t) - 1/2 Sin(2 t) + …



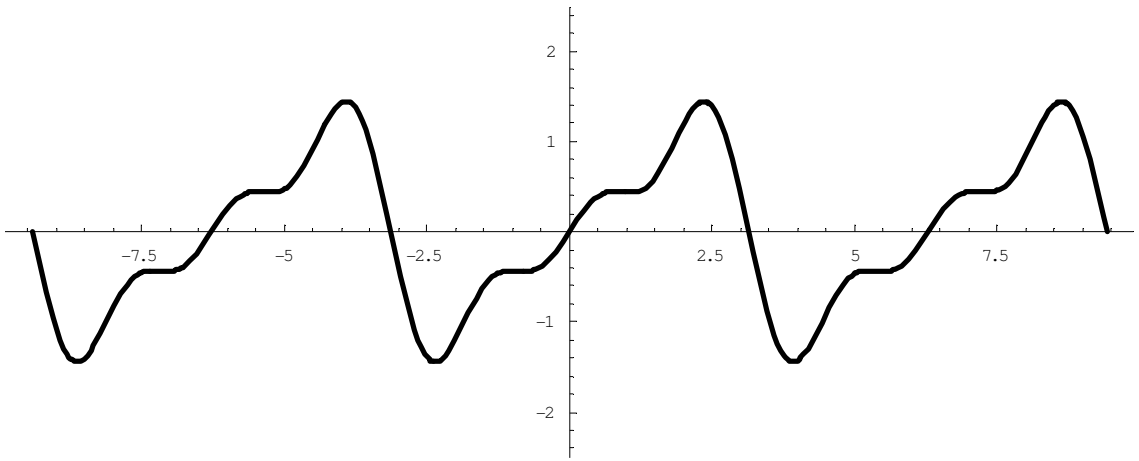
Fourier's Representation

1 Sin(t) - 1/2 Sin(2 t) +1/3 Sin(3 t) + …



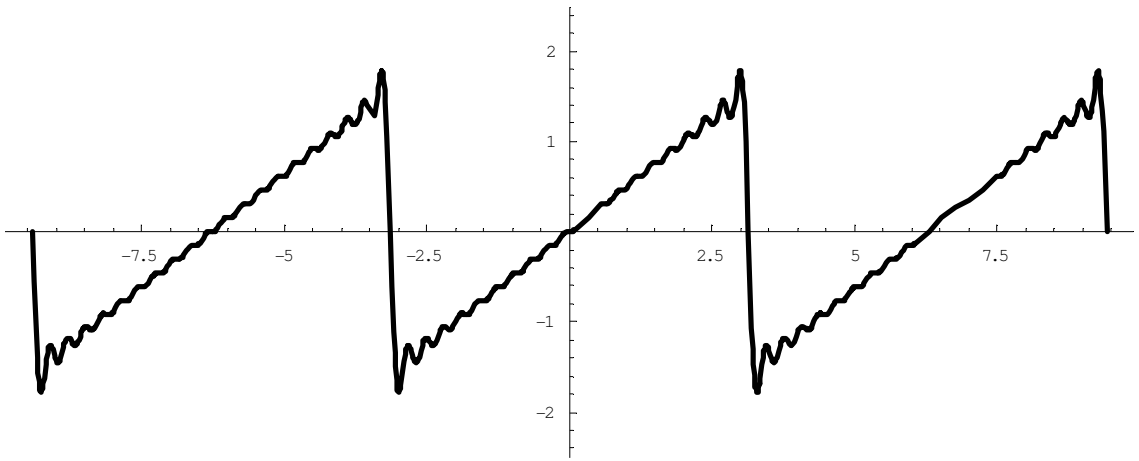
Fourier's Representation

$$1 \operatorname{Sin}(t) - 1/2 \operatorname{Sin}(2t) + 1/3 \operatorname{Sin}(3t) + \ldots$$



Fourier's Representation

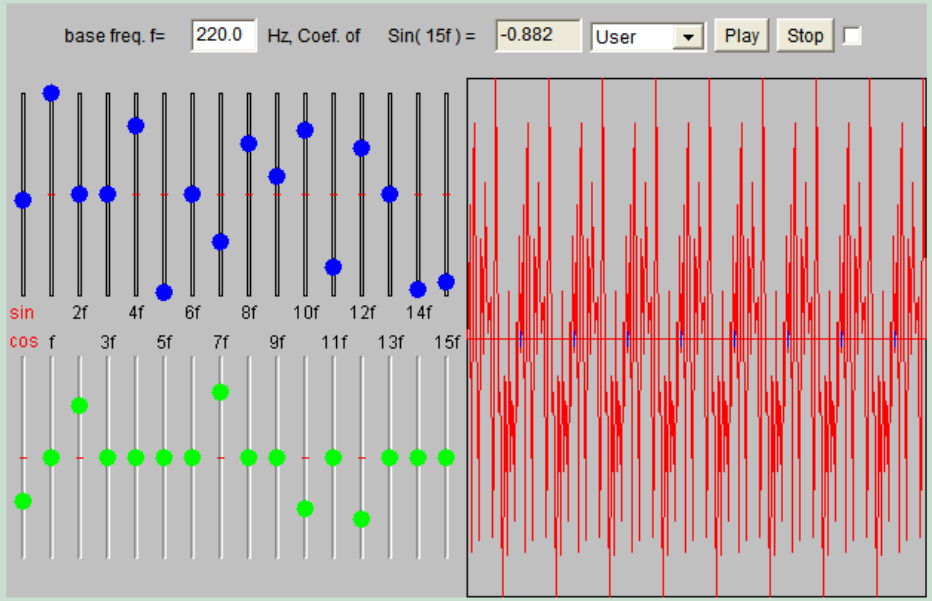
20'th degree Fourier expansion



Synthesize Music using Fourier Series!

http://www.phy.ntnu.edu.tw/ntnujava/viewtopic.php?t=33

- Similar to a music synthesizer



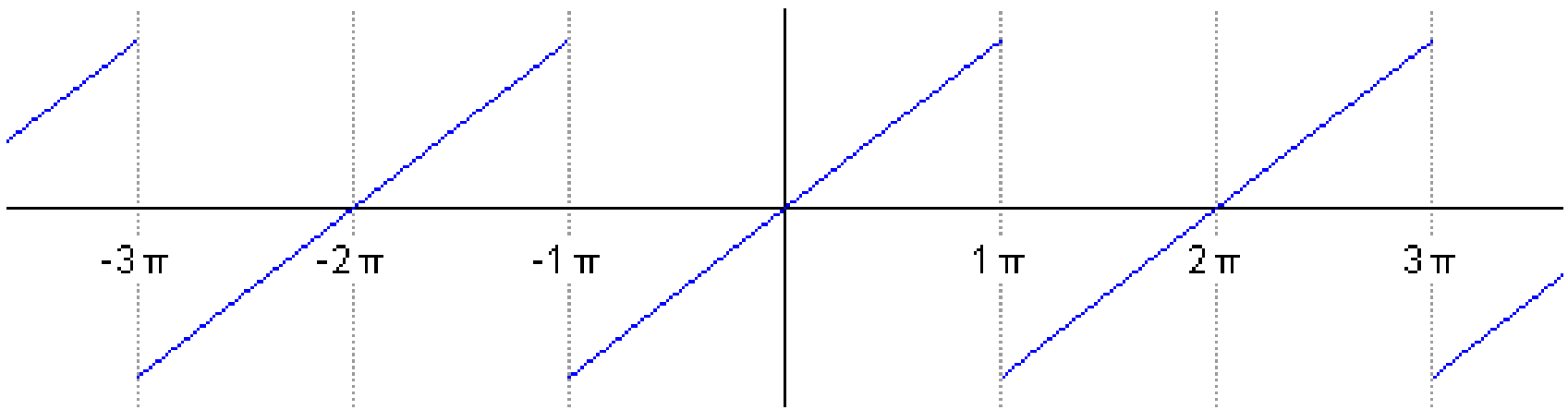
Click **get** button to fetch coefficients. Click **set** button to modify coefficients.

set	0f	f	2f	3f	4f	5f	6f	7f
sin	get	1.0	0.0	0.0	0.69	-0.98	0.0	-0.48
cos	-0.49	0.0	0.52	0.0	0.0	0.0	0.0	0.66
	8f	9f	10f	11f	12f	13f	14f	15f
sin	0.51	0.19	0.64	-0.73	0.47	0.0	-0.95	-0.88

Fourier series for periodic extension of nonperiodic function

Represent $f(t) = t$ for $|t| < \pi$ and 2π periodic

Just a ramp function repeated



How do you get the Coefficients for a given f ?

$$A_0/2 + A_1 \operatorname{Cos}(t) + A_2 \operatorname{Cos}(2t) + A_3 \operatorname{Cos}(3t) + \ldots$$

$$+ B_1 \operatorname{Sin}(t) + B_2 \operatorname{Sin}(2t) + B_3 \operatorname{Sin}(3t) + \ldots$$

Fourier's claim:

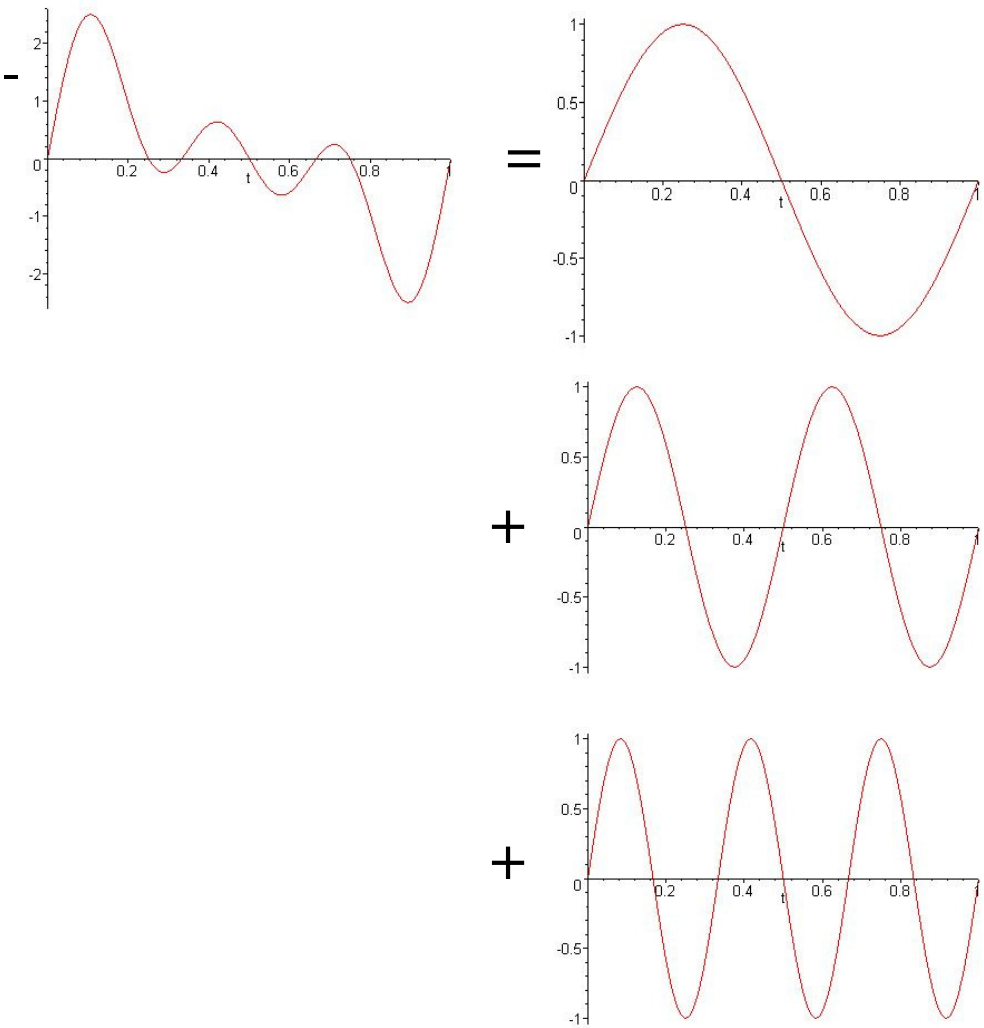
- *ANY* periodic function *f(t)* can be written this way (SYNTHESIS)
- The coefficients are uniquely determined by f: (ANALYSIS)

$$A_k = 1/2\pi \int_0^{2\pi} f(t) \cos(kt)\, dt$$

$$B_k = 1/2\pi \int_0^{2\pi} f(t) \sin(kt)\, dt$$

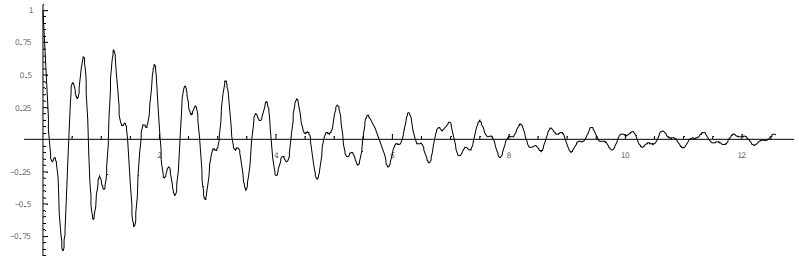
Fourier Analysis

- Def.: mathematical techniques for breaking up a signal into its components (sinusoids)
- Jean Baptiste Joseph Fourier (1768-1830)
- can represent any continuous periodic signal as a sum of sinusoidal waves

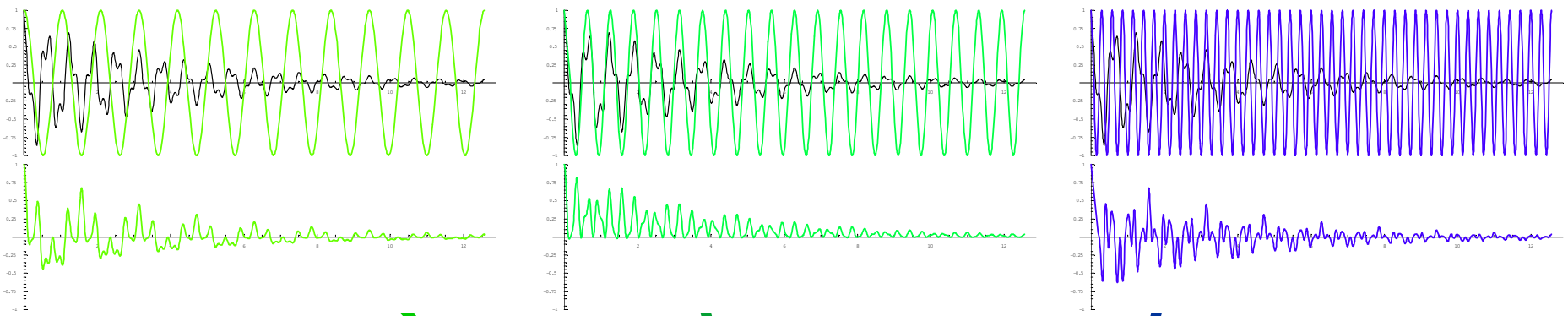


Fourier Analysis: match data with sinusoids

s(t)

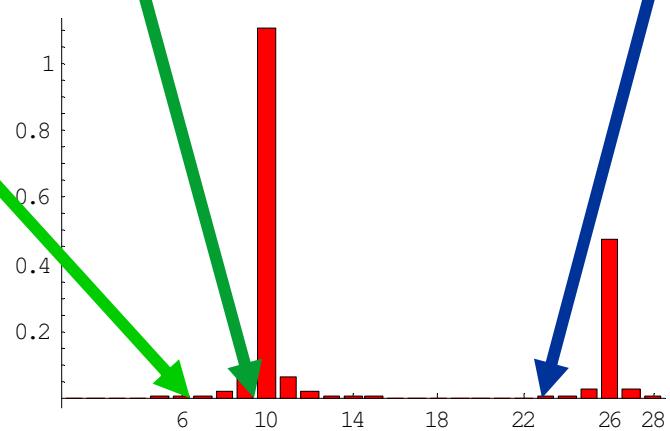


Time t



S[k]

$$= \int s(t)\ \cos(k\,t)\ dt$$



Frequency k

Complex Notation

For f, periodic with period p combine sines and cosines using identity $\exp(i\,\theta) = \cos\theta + i \sin\theta$

Fourier transform $f(t) \rightarrow F[k]$

$$F[k] = 1/p \int_0^p f(t) \cos(2\pi\, k\, t/p)\, dt$$

$$- i/p \int_0^p f(t) \sin(2\pi\, k\, t/p)\, dt$$

$$= 1/p \int_0^p f(t)\, e^{-2\pi i k t/p}\, dt$$

Inverse Fourier transform $F[k] \rightarrow f(t)$

$$f(t) = \sum F[k]\, e^{2\pi i k t/p}$$

DFT and its inverse for periodic discrete data

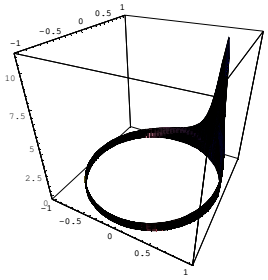
$$\Phi[k] = \sum_{n=0}^{N-1} \phi[n]\, e^{-2\pi i k n h/p} \qquad p = N h$$

$$= \sum_{n=0}^{N-1} \phi[n]\, e^{-2\pi i k n/N}$$

This is automatically periodic in k with period N

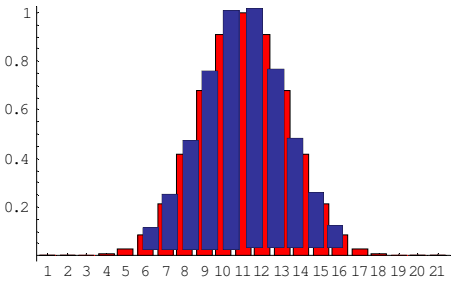
Inverse is like Fourier series, but with only p terms

Two PERIODIC time versions of Fourier



$$\int_0^p f(t)\, e^{-2\pi i k t/p}\, dt/p$$

$$\sum_{k \text{ in } \mathbb{Z}} F[k]\, e^{2\pi i k t/p}$$



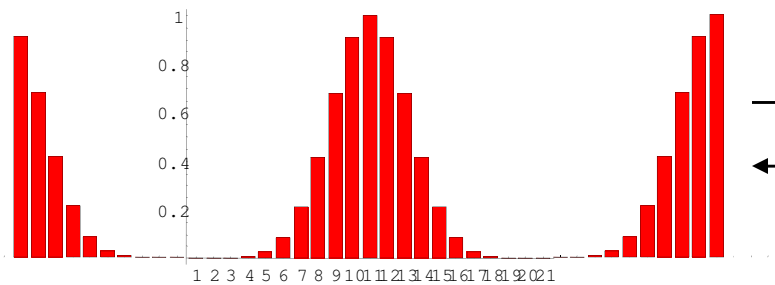
f(t), period p

F[k] , on $\mathbb{Z}$

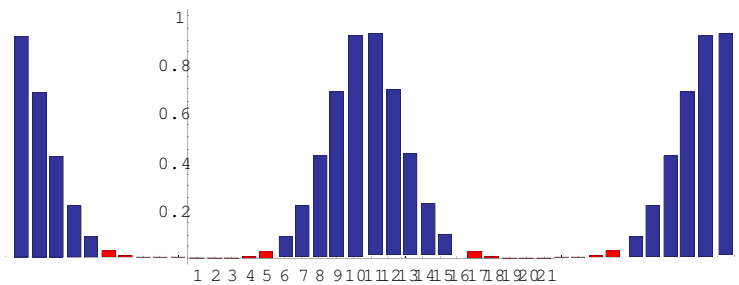
$\mathbf{S}_{p/N}$

$\mathbf{P}_N$

$$1/N \sum^{N-1} \gamma[k]\, e^{-2\pi i k m/N}$$



$k = 0$



$\gamma[k]$, on P_N

$$\sum_{k=0}^{N-1} \Gamma[k]\, e^{2\pi i m k/N}$$

Γ [m] on P_N

i.e. on $\mathbb{Z}$, Period N

i.e. on $\mathbb{Z}$, Period N

"time" domain

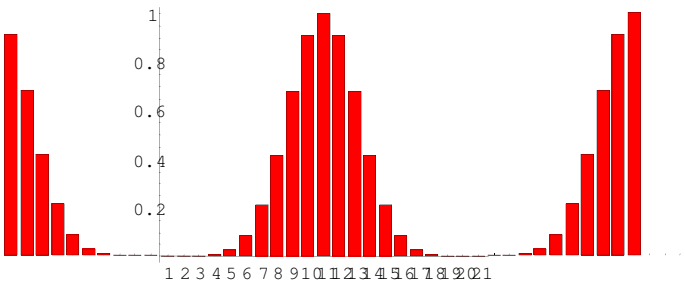
"frequency" domain

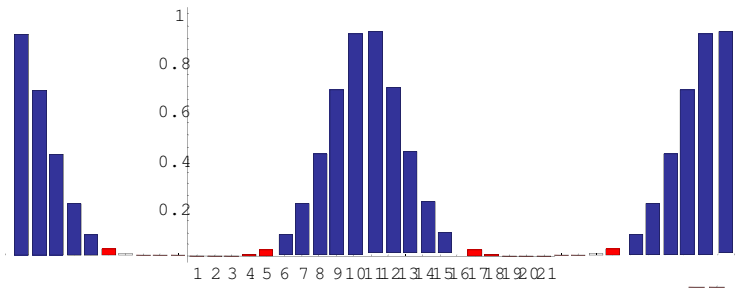
DFT: Discrete time periodic version of Fourier

"time" domain

"frequency" domain

$$1/N \sum_{k=0}^{N} \gamma[k]\, e^{-2\pi i k m/N} = \Gamma[m]$$





$$\gamma[k] = \sum_{k=0}^{N} \Gamma[k]\, e^{2\pi i m k/N}$$

$\gamma[k]$, on P_N

i.e. on Z, Period N

$\Gamma[m]$ on P_N

i.e. on Z, Period N