Eigenvalues and Eigenvectors

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Eigen Values of a Matrix

- Definition:
- A $N \times N$ matrix $A$ has an eigenvector $x$ (non-zero) with corresponding eigenvalue $\lambda$ if
  
  $$Ax = \lambda x$$

- This means
  
  $$Ax - \lambda x = 0$$
  
  $$\Rightarrow (A - \lambda I)x = 0$$

- If two numbers multiply to zero one of them is zero
- If a matrix vector product gives a zero vector, then either the vector is zero, or the matrix has zero determinant (is singular).
Solving for eigenvalues

- The zero vector is not an eigenvector (nothing special about $A0=0$)
- So we need $(A-\lambda I)x=0 \quad ||x||_2 \neq 0$
  \[ \text{det}(A-\lambda I)=0 \]
- Evaluating the determinant we get an $N$th degree polynomial equation, which can be solved for $N$ roots
  - Could be solved numerically using zero finding algorithms
- So a $N \times N$ matrix has $N$ eigenvalues

Characteristic Equation

- $Ax = \lambda x$ can be written as
  \[ (A-\lambda I)x = 0 \]
  which holds for $x \neq 0$, so $(A-\lambda I)$ is singular and
  \[ \text{det}(A-\lambda I) = 0 \]
- This is called the characteristic polynomial. If $A$ is $n \times n$ the polynomial is of degree $n$ and so $A$ has $n$ eigenvalues, counting multiplicities.
- Eigenvalues need not be distinct.
  - E.g. eigenvalues of identity matrix are given by solution of
    \[ (1-\lambda)^n = 0 \]
- So the matrix has $N$ repeated eigenvalues equal to 1
Example

\[
A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \quad A - \lambda I = \begin{pmatrix} 4 - \lambda & 3 \\ 1 & 2 - \lambda \end{pmatrix}
\]

\[\det(A - \lambda I) = 0 \quad \Rightarrow \quad (4 - \lambda)(2 - \lambda) - (1)(3) = 0\]

\[\lambda^2 - 6\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda - 5)(\lambda - 1) = 0\]

• Hence the two eigenvalues are 1 and 5.

Example (continued)

• Once we have the eigenvalues, the eigenvectors can be obtained by substituting back into \((A - \lambda I)x = 0\).

\[A - \lambda_1 I = \begin{pmatrix} 4 - 5 & 3 \\ 1 & 2 - 5 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\]

\[A - \lambda_2 I = \begin{pmatrix} 4 - 1 & 3 \\ 1 & 2 - 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\]

• Eigenvectors should give a zero vector when multiplied.

• This gives eigenvectors \((1 -1)^T\) and \((1 1/3)^T\)

• Note that we can scale the eigenvectors any way we want.
Assorted properties of eigenvalues & eigenvectors

• Shift eigenvalues of a matrix by \( \tau \).
  – Let \( Ax = \lambda \mathbf{x} \)
  – Add \(-\tau \mathbf{x}\) to both sides
    \[
    (A - \tau \mathbf{I})\mathbf{x} = (\lambda - \tau) \mathbf{x}
    \]
  – We get a new matrix
    \[
    B = (A - \tau \mathbf{I})
    \]
  – Shifted eigenvalue \((\lambda - \tau)\)
  – Same eigenvector \(\mathbf{x}\)

• Eigenvectors are not in general normalized:
  – If \(\mathbf{x}\) is an eigenvector so is \(\alpha \mathbf{x}\).
  – Often in software we may normalize eigenvectors to have \(||\mathbf{x}||_2 = 1\)

• The term *eigenvalue* is a partial translation of the German “eigenvert.” A complete translation would be something like “own value” or “characteristic value”.

Eigenvalues and eigenvector

• Recall a \(N \times N\) matrix maps \(N\) dimensional vectors to other \(N\) dimensional vectors
  – In general it maps elements in \(\mathbb{R}^N\) to other elements in \(\mathbb{R}^N\)

• Eigenvectors and eigenvalues provide basic information about this mapping
  – Identify special vectors which remain untransformed (or are just scaled)

• Important in many areas
  – Quantum mechanics – energy levels
  – Acoustics – fundamental frequencies of drums or columns
  – Stability theory – resonant frequencies or critical values of parameters
Eigen-value decomposition

- Represent the matrix in terms of its eigenvalues and eigenvectors
- A $N \times N$ matrix has $N$ eigenvalues and eigenvectors
- Write the $N$ equations
  \[ Ax_i = \lambda_i x_i \]
  - by stacking the vectors $x_i$ as columns of a matrix $X$ and the constants $\lambda_i$ along the diagonal of a matrix
- We get
  \[ AX = X\Lambda \]
  - If all eigenvectors are independent, then $X^{-1}$ exists, and so
  \[ X^{-1}AX = X^{-1}X\Lambda = \Lambda \]
  \[ A = X\Lambda X^{-1} \]
- This is the *eigenvalue decomposition* of a matrix $A$

Left and Right Eigenvectors

- So far we just talked about matrix products
  \[ Ax = \lambda x \]
- For a $N \times N$ matrix we can also define a left matrix product
  \[ y^t A = \nu \]
- So if we have
  \[ y^t A = \lambda y \]
  then $y$ is a left eigenvector of $A$
- If $A$ is symmetric $A = A^t$
- $A^t = x^t A^t = x^t A$
- So left and right eigenvectors of a symmetric matrix are the same
Symmetric Matrices

- A matrix is symmetric if its transpose is equal to itself.
- \( A \) is symmetric if \( A^\top = A \) For a complex matrix \( A^H = A \)
- Eigenvalues and Eigenvectors of a real symmetric (complex hermitian) matrix are real and eigenvectors are orthogonal.

Symmetric Matrices

- Eigenvalues and Eigenvectors of a real symmetric matrix are real. Its eigenvectors are orthogonal.
  \[ A \cdot X_R = X_R \cdot \text{diag}(\lambda_1 \ldots \lambda_N) \]
  \[ X_L \cdot A = \text{diag}(\lambda_1 \ldots \lambda_N) \cdot X_L \]
- Multiply first equation on left by \( X_L \), second on the right by \( X_R \), and subtract
  \[(X_L \cdot X_R) \cdot \text{diag}(\lambda_1 \ldots \lambda_N) = \text{diag}(\lambda_1 \ldots \lambda_N) \cdot (X_L \cdot X_R)\]
- matrix of dot products of the left and right eigenvectors commutes with the diagonal matrix of eigenvalues.
- Only matrices that commute with a diagonal matrix of distinct elements are themselves diagonal.
- So \( X_L \cdot X_R \) is diagonal
- So \( X_L \) and \( X_R \) are orthogonal to each other
- But \( X_L = X_R^\top \)
Positive Definite Matrices

• A complex matrix $A$ is positive definite if for every nonzero complex vector $x$ the quadratic form $x^H A x$ is real and:

$$x^H A x > 0$$

where $x^H$ denotes the conjugate transpose of $x$ (i.e., change the sign of the imaginary part of each component of $x$ and then transpose).

Eigenvalues of Positive Definite Matrices

• If $A$ is positive definite and $\lambda$ and $x$ are an eigenvalue/eigenvector pair, then:

$$A x = \lambda x \quad x^H A x = \lambda x^H x$$

• Since $x^H A x$ and $x^H x$ are both real and positive it follows that $\lambda$ is real and positive.
Properties of Positive Definite Matrices

• If A is a positive definite matrix then:
  – A is nonsingular.
  – The inverse of A is positive definite.
  – Gaussian elimination can be performed on A without pivoting.
  – The eigenvalues of A are positive.