Computational Methods
CMSC/AMSC/MAPL 460

Eigenvalues and Eigenvectors

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Eigen Values of a Matrix

• Already met eigenvalues and eigenvectors a few times in the course
• Here we will study them more formally
• Definition:
  • A $N \times N$ matrix $A$ has an eigenvector $x$ (non-zero) with corresponding eigenvalue $\lambda$ if
    \[ Ax = \lambda x \]
  • This means
    \[ Ax - \lambda x = 0 \quad \Rightarrow \quad (A - \lambda I)x = 0 \]
  • If two numbers multiply to zero one of them is zero
  • If a matrix vector product gives a zero vector, then either the vector is zero, or the matrix has zero determinant (is singular).
Solving for eigenvalues

- The zero vector is not an eigenvector (nothing special about $A0=0$)
- So we need $(A-\lambda I)x=0 \quad ||x||_2 \neq 0$
  \[ \det(A-\lambda I)=0 \]
- Evaluating the determinant we get an $N$th degree polynomial equation, which can be solved for $N$ roots
  - Could be solved numerically using zero finding algorithms
- So a $N \times N$ matrix has $N$ eigenvalues
- Of course eigenvalues need not be distinct.
- E.g. eigenvalues of identity matrix are given by solution of
  \[ (1-\lambda)^n =0 \]
- So the matrix has $N$ repeated eigenvalues equal to 1
Assorted properties of eigenvalues & eigenvectors

• Shift eigenvalues of a matrix by \( \tau \).
  – Let
    \[ \mathbf{A}\mathbf{x}=\lambda \mathbf{x} \]
  – Add \(-\tau \mathbf{x}\) to both sides
    \[ (\mathbf{A}-\tau \mathbf{I})\mathbf{x}=(\lambda-\tau) \mathbf{x} \]
  – We get a new matrix
    \[ \mathbf{B}= (\mathbf{A}-\tau \mathbf{I}) \]
  – Shifted eigenvalue \((\lambda-\tau)\)
  – Same eigenvector \(\mathbf{x}\)

• Eigenvectors are not in general normalized:
  – If \(\mathbf{x}\) is an eigenvector so is \(\alpha \mathbf{x}\).
  – Often in software we may normalize eigenvectors to have \(\|\mathbf{x}\|_2=1\)

• The term eigenvalue is a partial translation of the German “eigenvert.” A complete translation would be something like “own value” or “characteristic value”.
Eigenvalues and eigenvector

• Recall a $N \times N$ matrix maps $N$ dimensional vectors to other $N$ dimensional vectors
  – In general it maps elements in $\mathbb{R}^N$ to other elements in $\mathbb{R}^N$

• Eigenvectors and eigenvalues provide basic information about this mapping
  – Identify special vectors which remain untransformed (or just scaled)

• Important in many areas
  – Quantum mechanics – energy levels
  – Acoustics – fundamental frequencies of drums or columns
  – Stability theory – resonant frequencies or critical values of parameters
Eigen-value decomposition

- Represent the matrix in terms of its eigenvalues and eigenvectors
- A $N \times N$ matrix has $N$ eigenvalues and eigen vectors
- Write the $N$ equations
  \[ Ax_i = \lambda_i x_i \]
- by stacking the vectors $x_i$ as columns of a matrix $X$ and the constants $\lambda_i$ along the diagonal of a matrix
- We get
  \[ AX = X\Lambda \]
- If all eigenvectors are independent, then $X^{-1}$ exists, and so
  \[ X^{-1}AX = X^{-1}X\Lambda = \Lambda \]
  \[ A = X\Lambda X^{-1} \]
- This is the eigenvalue decomposition of a matrix $A$
Use of the eigenvalue decomposition

- Can use it to study the properties of $A$
- Recall condition number definition
  \[
  \text{cond}(A) = \kappa(A) = \|A\| \cdot \|A^{-1}\|
  \]
  \[
  = \frac{\text{maximum stretch}}{\text{maximum shrink}} = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}
  \]

- Natural frequencies of the matrix
- Powers of a matrix

\[
Ax = \lambda x
\]
\[
A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x
\]
\[
A^n x = \lambda^n x
\]

- Apply same idea to EVD

\[
A^n = X\Lambda^n X^{-1}
\]
Similar Matrices

- Two matrices $A$ and $B$ are said to be similar if it is possible to relate them as

$$B = T^{-1}AT \quad \quad TBT^{-1} = A$$

- Here $T$ is any non-singular matrix, which is the similarity transform matrix.

- Theorem: Similar matrices have the same eigenvalues and their eigen-vectors are related via the similarity transform.

- Proof. Let $(x, \lambda)$ be an eigen-pair for $A$. Then $Ax = \lambda x$.

Let $y = T^{-1}x$ and $x = Ty$

Then $Ax = \lambda x$.

Premultiply by $T^{-1}$ to get $T^{-1}ATT^{-1}x = \lambda T^{-1}x$

So $By = \lambda y$
EVD

- EVD is a similarity transform that takes $A$ to a diagonal matrix using a matrix of eigenvectors.
- Eigenvalue decomposition requires solving of a general polynomial equation.
  - Even if matrix has real entries eigenvalues can be complex
  - So can eigenvectors
- Eigenvectors provide a set of basis vectors in which the matrix becomes diagonal
Example from the book

- Let $A = \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix}$

This matrix was constructed in such a way that the characteristic polynomial factors nicely.

- $\det(A - \lambda I) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$

- Consequently the three eigenvalues are $\lambda_1 = 1, \lambda_2 = 2,$ and $\lambda_3 = 3,$ and $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

- The matrix of eigenvectors is

  $X = \begin{bmatrix} 1 & -4 & 7 \\ -3 & 9 & -49 \\ 0 & 1 & 91 \end{bmatrix}$

- It turns out that the inverse of $X$ also has integer entries.

  $X^{-1} = \begin{bmatrix} 130 & 43 & 133 \\ 27 & 9 & 28 \\ -3 & -1 & -3 \end{bmatrix}$

- These matrices provide the eigenvalue decomposition of our example $A = X\Lambda X^{-1}$
Eigshow

- Eigen values of $2 \times 2$ matrix represent transformations in the plane
- Ideas of symmetry
Left and Right Eigenvectors

• So far we just talked about matrix products \( Ax = \lambda x \)

• For a \( N \times N \) matrix we can also define a left matrix product

\[ y^t A = v \]

• So if we have

\[ y^t A = \lambda y \]

then \( y \) is a left eigenvector of \( A \)

• If \( A \) is symmetric \( A = A^t \)

• \( (Ax)^t = x^t A^t = x^t A \)

• So left and right eigenvectors of a symmetric matrix are the same
Symmetric Matrices

- A matrix is symmetric if its transpose is equal to itself
- \( A \) is symmetric if \( A^t = A \)
- Eigenvalues and Eigenvectors of a real symmetric matrix are real. Its eigenvectors are orthogonal.

\[
A \cdot X_R = X_R \cdot \text{diag}(\lambda_1 \ldots \lambda_N)
\]

\[
X_L \cdot A = \text{diag}(\lambda_1 \ldots \lambda_N) \cdot X_L
\]

- Multiply first equation on left by \( X_L \), second on the right by \( X_R \), and subtract

\[
(X_L \cdot X_R) \cdot \text{diag}(\lambda_1 \ldots \lambda_N) = \text{diag}(\lambda_1 \ldots \lambda_N) \cdot (X_L \cdot X_R)
\]

- Matrix of dot products of the left and right eigenvectors commutes with the diagonal matrix of eigenvalues.

- Only matrices that commute with a diagonal matrix of distinct elements are themselves diagonal.

- So \( X_L \cdot X_R \) is diagonal