

Computational Methods
CMSC/AMSC/MAPL 460

Eigenvalues and Eigenvectors

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Eigen Values of a Matrix

- Already met eigenvalues and eigenvectors a few times in the course
- Here we will study them more formally
- Definition:
- A $N \times N$ matrix \mathbf{A} has an eigenvector \mathbf{x} (non-zero) with corresponding eigenvalue λ if

$$\mathbf{Ax} = \lambda \mathbf{x}$$

- This means

$$\mathbf{Ax} - \lambda \mathbf{x} = 0 \qquad (\mathbf{A} - \lambda \mathbf{I}) \mathbf{x} = 0$$

- If two numbers multiply to zero one of them is zero
- If a matrix vector product gives a zero vector, then either the vector is zero, or the matrix has zero determinant (is singular).

Solving for eigenvalues

- The zero vector is not an eigenvector (nothing special about $\mathbf{A}\mathbf{0}=\mathbf{0}$)

- So we need $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ $\|\mathbf{x}\|_2 \neq 0$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

- Evaluating the determinant we get an N th degree polynomial equation, which can be solved for N roots
 - Could be solved numerically using zero finding algorithms

- So a $N \times N$ matrix has N eigenvalues

- Of course eigenvalues need not be distinct.

- E.g. eigenvalues of identity matrix are given by solution of

$$(1-\lambda)^n = 0$$

- So the matrix has N repeated eigenvalues equal to 1

Assorted properties of eigenvalues & eigenvectors

- Shift eigenvalues of a matrix by τ .

- Let

$$\mathbf{A}\mathbf{x}=\lambda \mathbf{x}$$

- Add $-\tau \mathbf{x}$ to both sides

$$(\mathbf{A}-\tau \mathbf{I})\mathbf{x}=(\lambda-\tau) \mathbf{x}$$

- We get a new matrix

$$\mathbf{B}=(\mathbf{A}-\tau \mathbf{I})$$

- Shifted eigenvalue $(\lambda-\tau)$

- Same eigenvector \mathbf{x}

- Eigenvectors are not in general normalized:

- If \mathbf{x} is an eigenvector so is $\alpha \mathbf{x}$.

- Often in software we may normalize eigenvectors to have $\|\mathbf{x}\|_2=1$

- The term *eigenvalue* is a partial translation of the German “eigenvert.” A complete translation would be something like “own value” or “characteristic value” .

Eigenvalues and eigenvector

- Recall a $N \times N$ matrix maps N dimensional vectors to other N dimensional vectors
 - In general it maps elements in \mathbb{R}^N to other elements in \mathbb{R}^N
- Eigenvectors and eigenvalues provide basic information about this mapping
 - Identify special vectors which remain untransformed (or just scaled)
- Important in many areas
 - Quantum mechanics – energy levels
 - Acoustics – fundamental frequencies of drums or columns
 - Stability theory – resonant frequencies or critical values of parameters

Eigen-value decomposition

- Represent the matrix in terms of its eigenvalues and eigenvectors
- A $N \times N$ matrix has N eigenvalues and eigen vectors
- Write the N equations

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

- by stacking the vectors \mathbf{x}_i as columns of a matrix \mathbf{X} and the constants λ_i along the diagonal of a matrix
- We get

$$\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{\Lambda}$$

- If all eigenvectors are independent, then \mathbf{X}^{-1} exists, and so

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \mathbf{X}^{-1}\mathbf{X}\mathbf{\Lambda} = \mathbf{\Lambda}$$

$$\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$$

- This is the **eigenvalue decomposition** of a matrix \mathbf{A}

Use of the eigenvalue decomposition

- Can use it to study the properties of A
- Recall condition number definition

$$\text{cond}(A) = \kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$$= \frac{\text{maximum stretch}}{\text{maximum shrink}} = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}$$

- Natural frequencies of the matrix
- Powers of a matrix

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{A}(\mathbf{Ax}) = \mathbf{A}(\lambda \mathbf{x}) = \lambda \mathbf{Ax} = \lambda^2 \mathbf{x}$$

$$\mathbf{A}^n \mathbf{x} = \lambda^n \mathbf{x}$$

- Apply same idea to EVD

$$\mathbf{A}^n = \mathbf{X}\mathbf{\Lambda}^n\mathbf{X}^{-1}$$

Similar Matrices

- Two matrices \mathbf{A} and \mathbf{B} are said to be similar if it is possible to relate them as

$$\mathbf{B} = \mathbf{T}^{-1} \mathbf{A} \mathbf{T} \qquad \mathbf{T} \mathbf{B} \mathbf{T}^{-1} = \mathbf{A}$$

- Here \mathbf{T} is any non singular matrix, which is the similarity transform matrix
- Theorem: Similar matrices have the same eigenvalues and their eigen-vectors are related via the similarity transform.
- Proof. Let (\mathbf{x}, λ) be an eigen-pair for \mathbf{A} . Then $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.

Let $\mathbf{y} = \mathbf{T}^{-1} \mathbf{x}$ and $\mathbf{x} = \mathbf{T} \mathbf{y}$

Then $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.

Premultiply by \mathbf{T}^{-1} to get $\mathbf{T}^{-1} \mathbf{A} \mathbf{T} \mathbf{T}^{-1} \mathbf{x} = \lambda \mathbf{T}^{-1} \mathbf{x}$

So $\mathbf{B}\mathbf{y} = \lambda \mathbf{y}$

EVD

- EVD is a similarity transform that takes A to a diagonal matrix using a matrix of eigenvectors.
- Eigenvalue decomposition requires solving of a general polynomial equation.
 - Even if matrix has real entries eigenvalues can be complex
 - So can eigenvectors
- Eigenvectors provide a set of basis vectors in which the matrix becomes diagonal

Example from the book

- Let $A = [-149 \ -50 \ -154; 537 \ 180 \ 546; -27 \ -9 \ -25]$

This matrix was constructed in such a way that the characteristic polynomial factors nicely.

- $\det(A - \lambda I) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$
- Consequently the three eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$, and $\Lambda = [1 \ 0 \ 0; 0 \ 2 \ 0; 0 \ 0 \ 3]$
- The matrix of eigenvectors is

$$X = [1 \ -4 \ 7 ; -3 \ 9 \ -49; 0 \ 1 \ 91]$$

- It turns out that the inverse of X also has integer entries.

$$X^{-1} = [130 \ 43 \ 133 ; 27 \ 9 \ 28 ; -3 \ -1 \ -3]$$

- These matrices provide the eigenvalue decomposition of our example $A = X\Lambda X^{-1}$

Eigshow

- Eigen values of 2×2 matrix represent transformations in the plane
- Ideas of symmetry

Left and Right Eigenvectors

- So far we just talked about matrix products

$$Ax = \lambda x$$

- For a $N \times N$ matrix we can also define a left matrix product

$$y^t A = \lambda y$$

- So if we have

$$y^t A = \lambda y$$

then y is a left eigenvector of A

- If A is symmetric $A = A^t$
- $(Ax)^t = x^t A^t = x^t A$
- So left and right eigenvectors of a symmetric matrix are the same

Symmetric Matrices

- A matrix is symmetric if its transpose is equal to itself
- \mathbf{A} is symmetric if $\mathbf{A}^t = \mathbf{A}$
- Eigenvalues and Eigenvectors of a real symmetric matrix are real. Its eigenvectors are orthogonal.

$$\mathbf{A} \cdot \mathbf{X}_R = \mathbf{X}_R \cdot \text{diag}(\lambda_1 \dots \lambda_N)$$

$$\mathbf{X}_L \cdot \mathbf{A} = \text{diag}(\lambda_1 \dots \lambda_N) \cdot \mathbf{X}_L$$

- Multiply first equation on left by \mathbf{X}_L , second on the right by \mathbf{X}_R , and subtract

$$(\mathbf{X}_L \cdot \mathbf{X}_R) \cdot \text{diag}(\lambda_1 \dots \lambda_N) = \text{diag}(\lambda_1 \dots \lambda_N) \cdot (\mathbf{X}_L \cdot \mathbf{X}_R)$$

- matrix of dot products of the left and right eigenvectors commutes with the diagonal matrix of eigenvalues.
- Only matrices that commute with a diagonal matrix *of distinct elements* are themselves diagonal.
- So $\mathbf{X}_L \cdot \mathbf{X}_R$ is diagonal